الحل الصريح لمعادلة القطع المكافئ

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مقدمة

لتقريب معادلة القطع المكافي في النطاق غير المنظم استعمل ميلن ١١ نقطة، ولكن هذه الطريقة معقدة في التطبيق. المؤلفان ١ و ٤ استعملا صيغة الخمسة نقاط وهي طريقة اسهل ولكن اخطاءها كبيرة . مع ذلك ففي هذا العمل طريقة مثلى نستعمل فيها صيغة ٩ نقاط لمؤثر لابلاس في النطاق غير المنتظم وهي طريقة اسهل بكثير من الطريقة التي استعملها ميلن ولها تقريب جيد بالمقارنة مع الحل التحليلي.

Explicit solution of Parabolic Equation

I- The operator $\overline{K}$

Let H and X be partial differential operators associated with Laplace differential operator in two dimensions defined by:

$$HF(x,y) = F(x+h,y) + F(x-h,y) + F(x,y+h) + F(x,y-h) - 4F(x,y)$$

$$XF(x,y) = (1/2)[F(x+h,y+h) + F(x-h,y+h) + F(x-h,y-h) + F(x+h,y-h) - 4F(x,y)]$$

each of them approximate ∇^2 in regular domain [4], i.e.

$$h^2\nabla^2 u = Hu + O(h^4)$$

$$h^2\nabla^2 u = Xu + O(h^4)$$

Another operator depending on nine points given by [4], [5]:

$h^2\nabla^2 u = Ku + O(h^6)$, where K=4H+2X.

is more accurate than both H and X.

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In irregular domain $\overline{H}$ and $\overline{X}$ where presented in [3], as follows:

$$(1)\overline{H}u=\frac{2}{h_1+h_3}\left\{\frac{u(x+h_1,y)-u(x,y)}{h_1}+\frac{u(x-h_3,y)-u(x,y)}{h_3}\right\}+\frac{2}{h_2+h_4}\left\{\frac{u(x,y+h_2)-u(x,y)}{h_2}+\frac{u(x,y-h_4)-u(x,y)}{h_4}\right\}$$

$$(2)\overline{X}u=\frac{1}{h_5+h_7}\left\{\frac{u(x+h_5,y+h_5)-u(x,y)}{h_5}+\frac{u(x-h_7,y-h_7)-u(x,y)}{h_7}\right\}+\frac{1}{h_6+h_8}\left\{\frac{u(x-h_6,y+h_6)-u(x,y)}{h_6}+\frac{u(x+h_8,y-h_8)-u(x,y)}{h_8}\right\}$$

In [1] and [2] $\overline{K}$ has been derived in which seven points from nine points that $\overline{K}$ depend on may be non-nodel, $\overline{K}$ given by: $\overline{K}=4\overline{H}+2\overline{X}$, i.e.

$$(3)\overline{K}u=\nabla^2 u=\sum_{i=0}^{8}\alpha_i u_i \text{ , where}$$

$$(4)\alpha_1=\frac{8}{h^2 s_1(s_1+s_3)},\ \alpha_2=\frac{8}{h^2 s_2(s_2+s_4)},\ \alpha_3=\frac{8}{h^2 s_3(s_3+s_1)},$$

$$\alpha_4=\frac{8}{h^2 s_4(s_4+s_2)},\ \alpha_5=\frac{2}{h^2 s_5(s_5+s_7)},\ \alpha_6=\frac{2}{h^2 s_6(s_6+s_8)},\ \alpha_7=\frac{2}{h^2 s_7(s_7+s_5)},$$

$$\alpha_8=\frac{2}{h^2 s_8(s_8+s_6)} \text{ and}$$

$\alpha_0=-\sum_{i=0}^{8}\alpha_i$, where h is $h_i=s_i h,\ \ 0<s_i\le 1$, i=1,2,3,4,5,6,7,8 and h is the step length of the grid.

II- Parabolic Equation

By Differentiation the Parabolic equation $U_t=C^2\nabla^2 U$ we get, [4]:

$$(5)U_t^m=C^{2m}\nabla^{2m}U \text{ , m=1,2,3,... .}$$

Using Taylor series,

$$(6)U(x,y,t+k)-U(x,y,t)=kU_t+\frac{k^2U_{tt}}{2!}+\frac{k^3U_{ttt}}{3!}+....=(\frac{1}{6})h^2\nabla^2U+(\frac{1}{72})h^4\nabla^4U+(\frac{1}{296})h^6\nabla^6U+....$$

$$=kU_t+(\frac{k^2}{2!})U_{tt}+(\frac{k^3}{3!})U_{ttt}+....$$

where $k=\frac{h^2}{6C^2}$, substituting $\overline{K}=h^2\nabla^2$ in (6) we get :

$$(7)U(x,y,t+k)-U(x,y,t)=(\frac{kC^2}{6})\overline{K}U+(\frac{k^2C^4}{72})\overline{K}^2U+(\frac{k^3C^6}{1296})\overline{K}^3U+....$$

In actual computation all terms un the right hand side are dropped except the first term hence,

$$(8)U(x,y,t+k)=U+(\frac{kC^2}{6})\overline{K}U=\left\{\frac{kC^2\overline{K}+6)}{6}\right\}U$$, where

$\overline{K}=$

α_6	α_2	α_5
α_3	α_0	α_1
α_7	α_4	α_8

where $\alpha_i=1,2,3,4,5,6,7,8$ as in equation (3).

The dropped terms in (7) which is equation (8) can be used to estimate the magnitude of truncation error of that equation.

III-Implementation:

The solution of $U_t=C^2\nabla^2U$ in the quadrant of the circle bounded by x=0, y=0, $x^2+y^2=36$, with boundary conditions U=0, and the initial values as shown in the following figure.

Solution: The mesh length equal one, applying $\overline{K}$ operator, we compute the values at t=k and t=2k as shown below for the corresponding placed points in fig.1 , and we take the value of $k=1/(6C^2)$, the tables 1 and 2 shows the solutions for t=k and t=2k resp. Tables 3 and 4 shows that the difference between our results and the analytical solution in polar coordinate resp.[4].

Table –1- (t=k)

0	0.0883736	0.1031124	0.037785		
0	0.1894806	0.2857445	0.2405551	0.0824154	
0	0.2496972	0.3973806	0.3890722	0.2405578	0.037953
0	0.2358972	0.3844084	0.3973806	0.285755	0.1031338
0	0.1431972	0.2358972	0.2496972	0.189483	0.0883476
0	0	0	0	0	0

Table-2- (t=2k)

0	0.0771379	0.08857237	0.03631343		
0	0.1674290	0.25145140	0.20946760	0.07312851	
0	0.2210095	0.35161930	0.34366050	0.20946810	0.03629403
0	0.2088012	0.34024220	0.35162060	0.25145770	0.08858176
0	0.1267428	0.20880120	0.22101010	0.16743330	0.07714548
0	0	0	0	0	0

Table-3- (t=k)

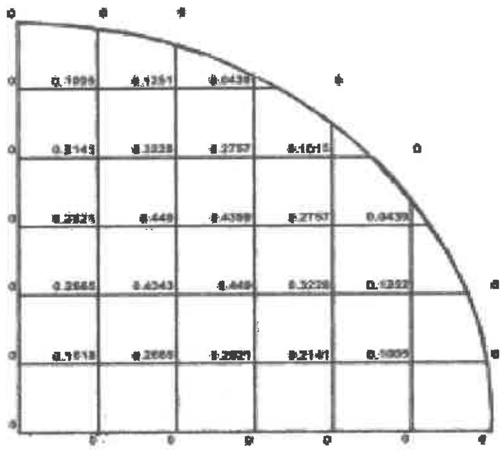
0	0.0005	0.0075	0.0081		
0	0.00001	0.00004	0.0034	0.0073	
0	0.000002	0.00001	0.00007	0.0034	0.0081
0	0.000002	0.00008	0.00001	0.00005	0.0075
0	0.000002	0.000002	0.000002	0.00001	0.0005
0	0	0	0	0	0

Table-4- (t=2k)

0	0.0015	0.0094	0.0019		
0	0.0002	0.0014	0.0065	0.0064	
0	0.000009	0.00008	0.0009	0.0060	0.0018
0	0.000001	0.00004	0.00001	0.0002	0.0094
0	0.000040	0.000001	0.00001	0.0002	0.0015
0	0	0	0	0	0

Conclusion:

The method implemented in this work is simpler than those methods, which use 11-points and the approximation to the solution is similar, as well as this solution is more accurate than those methods using five points, i.e. $\overline{H}$ and $\overline{X}$.



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Abstract: Milne [11] applied 11-points formula in irregular domain to approximate the solution of Parabolic equation, But this method is complicated in application. Others [1], [4], applied 5-points formula which is simpler methods but with considerable errors. However in this work an optimal method has been used nine points formula of Laplace operator in irregular domain which is considerably simple method in comparison with Milne's method and has a good approximations with respect to the analytical solution.