FUZZY OPERATOR

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Abstract:

In this paper, we introduce an algebra of special kind of fuzzy numbers(which we call it finite level fuzzy numbers) and also we introduce three kinds of fuzzy functions [4]. The integral and differential of fuzzy function was defined in the literature [4,5], but we generalize these concepts (fuzzy integral and fuzzy differential) by constructing the functional of fuzzy function. Using Zadeh's extension principle, the operator of fuzzy function is defined. Fuzzy Laplace transform is considered as a fuzzy operator, and the problem of linearity is addressed. Finally, we show that the convolution theorem can be extended in fuzzy theory.

1. Introduction

The concept of fuzzy sets as introduced by Zadeh was proved to be useful in modeling inexact systems[2], where subjective judgments and the non statistical nature of inexactness make classical methods of probability theory unsuitable. The concept of fuzzy integral as developed in [14] and [6] has been used in the problems of optimization with inexact constraints[12,7]. D.Dubois and H.Prade in [5] deals with differentiation of ordinary functions at a fuzzy point and differentiation of fuzzy-valued functions at a non fuzzy point. In both cases the concept of differentiation is extended. The authors also introduced the notions of fuzzy numbers and defined its basic operations [3]. R.Goeotschel, A.Kaufmann, M.Gupta and G.Zhang[1,8,9,10,11] have done much work about fuzzy numbers.

Throughout this paper, $\widetilde{F}, \widetilde{P}(X)$, R^X and $F(R)$ will denote a fuzzy function, a set of all positive fuzzy subsets of X, and a set of all functions from X to R, set of fuzzy numbers defined on R, respectively.

2.Finite Level Fuzzy Numbers

Fuzzy numbers have many forms in the real world, and there are some special classes of fuzzy numbers for which computations of their sum, for example is easy. One such class is triangular fuzzy number, another one is that of trapezoidal fuzzy number. But both of these kinds dose not have a group structure, i.e, the multiplication of two triangular (trapezoidal) fuzzy numbers is not a triangular (trapezoidal) fuzzy number, and for this reason we construct a new type of fuzzy numbers. Using

According to the extension principle [4], a binary operation * can be extended into (*) to combine two fuzzy numbers A and B. Moreover , if μ_A and μ_B are the membership functions of A and B assumed to be continuous functions on R:

$$\mu_{A(*)B}(z) = \underset{z=x*y}{Sup}\, Min(\mu_A(x), \mu_B(y)) \qquad (2.1)$$

Now, the construction of the finite level fuzzy numbers will be as follows:

Given n , N be two positive integers n < N, and $\alpha_1, \alpha_2, \cdots, \alpha_N \in [0,1]$ such that

$$\alpha_1 < \alpha_2 < \cdots < \alpha_{n-1} < \alpha_n = 1$$

$$\alpha_N < \alpha_{N-1} < \cdots < \alpha_{n+1} < \alpha_n = 1$$

Let $F(\Re_N)$ be the set of all fuzzy numbers $A = \{(x_i, \alpha_i)\}_N$ defined on R, such that

$$x_1 < x_2 < \cdots < x_N.$$

Any fuzzy set A must satisfies three conditions (convex, normalized, and its membership functions is defined on R and piecewise continuous) such that A can be considered as a fuzzy number. If A is not convex, we can construct a convex fuzzy set from a non-convex fuzzy set A, by generating a convex hull of A, as follows:

Given a fuzzy set $A \in \widetilde{P}(X)$ which is not convex, then a convex hull <A> of the fuzzy set A is defined by

$$< A >= \bigcap \{\widetilde{x} | A \subseteq \widetilde{x}, \widetilde{x} \text{ is convex fuzzy set } \}.$$

In another word, <A> is the least convex fuzzy set which contains A.

$$\mu_{<A>}(x) = \begin{cases} \alpha_1 & x_1 \le x < x_2 \\ \alpha_2 & x_2 \le x < x_3 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \alpha_{N-1} & x_{N-2} < x \le x_{N-1} \\ \alpha_N & x_{N-1} < x \le x_N \end{cases}$$

So the set $F(R_N)$ now can be considered as a set of fuzzy numbers. The operations of this type of fuzzy numbers can be defined by:

Let A, B $\in F(\Re_N)$ such that $A = \{(x_i, \alpha_i)\}_N$, $B = \{(y_i, \alpha_i)\}_N$
According to equation (2.1) we have

$$\begin{aligned} \mu_{A*B}(z) &= Sup\{Min(\mu_A(x), \mu_B(y)) | z = x * y\} \\ &= Max\{Min(\alpha_i, \alpha_j) | z = x_i * y_i\} \end{aligned} \quad (2.2)$$

If we perform the (*) operation between A and B we will get the following table:

*	y_1	y_2	·	·	·	y_n	·	·	·	y_N
x_1	α_1	α_1	·	·	·	α_1				
x_2	α_1	α_2	·	·	·	α_2	$[Min\{\alpha_i, \alpha_j\}]$			
·	·									
·	·									
x_n	·									
·	α_1	α_n	·	·	·	1				
·	$[Min\{\alpha_i, \alpha_j\}]$									
·						α_{N-1}		α_{N-1}		α_{N-1}
x_N						α_N		α_N		α_N

Now, from this table it is clear that the convex hull of A∗B is:

$$\mu_{<A*B>}(z) = \begin{cases} \alpha_1 & for \quad x_1 * y_1 \le z < x_2 * y_2 \\ \alpha_2 & for \quad x_2 * y_2 \le z < x_3 * y_3 \\ \cdot & \\ \cdot & \\ 1 & for \quad z = x_n * y_n \\ \alpha_{n+1} & for \quad x_n * y_n < z \le x_{n+1} * y_{n+1} \\ \cdot & \\ \cdot & \\ \alpha_N & for \quad x_{N-1} * y_{N-1} < z \le x_N * y_N \end{cases} \quad (2.3)$$

So, according to equation (2.3), equation (2.2) in this case can be written as

$$\mu_{A*B}(z) = \alpha_i \quad if \quad z = x_i * y_i \tag{2.4}$$

where

$$\mu_{A*B}(z) = 1 \quad if \quad z = x_n * y_n$$

and

$$z_1 < z_2 < \cdots < z_n = x_n * y_n$$

and

$$x_n * y_n = z_n < z_{n+1} < \cdots < z_N$$

so $\widetilde{Z} = \{(z_i, \alpha_i)\}_N$ is fuzzy number and $\widetilde{Z} \in F(\Re_N)$.

Example 2.1

Let A={(2,0.2),(3,0.4),(5,1),(7,0.5)},
B={(4,0.2),(6,0.4),(9,1),(10,0.5)}

And we want to compute A + B and A · B,

We have , $x_i = 2,3,5,7,\ y_i = 4,6,9,10$

It is clear that $z_i = x_i + y_i = 6,9,14,17$, $\alpha_i = 0.2,0.4,1,0.5 \quad i = 1,2,3,4$

So $\widetilde{Z} = \{(6,0.2),(9,0.4),(14,1),(17,0.5)\}$.

Now for A · B, we have

$$z_i = x_i \cdot y_i = 8,18,45,70,\ \alpha_i = 0.2,0.4,1,0.5 \quad i = 1,2,3,4$$

and

$$\widetilde{Z} = \{(8,0.2),(18,0.4),(45,1),(70,0.5)\}.$$

2.Fuzzy function

The extension principle, introduced by Zadeh [15], is one of the most important tools of fuzzy sets theory. Using this principle . any mathematical relationship between non-fuzzy elements can be fitted to deal with fuzzy quantities.

Definition 3.1 (Extension of Fuzzy Set) If f: X→Y, and A be a fuzzy set defined on X, then we can obtain a fuzzy set f(A) in Y by f and A,

$$\forall\, y \in Y,\ \mu_{f(A)}(y) = \begin{cases} Sup\ \{\mu_A(x) & if\ \ f^{-1}(y) \neq \phi, \forall x \in X, y = f(x) \\ 0 & if\ \ f^{-1}(y) = \phi \end{cases} \tag{3.1}$$

Fuzzy function can be classified into three basic kinds according to where fuzziness occurs.

(1) Crisp function with fuzzy constraint.
(2) Crisp function which propagates the fuzziness of independent variable to dependent variable.

(3) Function that it self fuzzy. This fuzzy function blurs the image of a crisp independent variable. For our work we need the third kind of fuzzy function.

Definition 3.2 A fuzzy mapping $\widetilde{F}$ is a mapping from X to the set of non-empty fuzzy sets on X, namely $\widetilde{P}(X)$, In other words, to each $x \in X$, corresponds a fuzzy set $\widetilde{F}(x)$ defined on X, whose membership function is $\mu_{\widetilde{F}(x)}$ and

$$\mu_{\widetilde{F}(x)} : X \to I \equiv [0,1]$$

A fuzzy set of mapping F can be constructed in the following way,

Define a function $F : X \to \widetilde{P}(X)$ such that $\mu_F : \Re^X \to I$,(where $\Re^X$ is the set of all functions f: X→ℜ)

$$\mu_F(f) = Inf\{\mu_{\widetilde{F}(x)}(f(x)) | x \in X\} \tag{3.2}$$

The converse of the above construction can be done as expressed in the following definition.

Definition 3.3 Given a fuzzy set of mappings $F : X \to \widetilde{P}(X)$ such that $\mu_F : \Re^X \to I$, we can construct a fuzzy function $\widetilde{F} : X \to \widetilde{P}(X)$ such that $\widetilde{F}(x)$ is a fuzzy set, by the follows:

$$\mu_{\widetilde{F}(x)}(y) = \begin{cases} Sup\{\mu_F(f) | x \in f^{-1}(y)\} & when \ \ f^{-1}(y) \neq \phi \\ 0 & when \ \ f^{-1}(y) = \phi \end{cases} \tag{3.3}$$

Definition 3.4. Let T be a fuzzy set such that T: X →ℜ, then T will be finite if $Supp(T) = \{x_i\}_n$. In another word, $T = \{(x_i, \alpha_i)\}_n$ where $\mu_T(x_i) = \alpha_i > 0$.

Definition 3.5. According to Def. 3.4 , if a fuzzy mapping $\widetilde{F} : X \to \widetilde{P}(X)$ is finite , then $\widetilde{F}$ can be written as

$$\widetilde{F}(x) = \{(f_i(x), \alpha_i)\}_n \tag{3.4}$$

Any fuzzy set of mappings F, constructed from $\widetilde{F}$ also will be finite, and

$$\mu_F(f) = Inf\{\mu_{\widetilde{F}(x)}(f(x)) | x \in X\} = \alpha_i \quad iff \quad f = f_i$$

This implies that $F = \{(f_i, \alpha_i)\}_n$.

Now, if given a finite set of mappings $F = \{(f_i, \alpha_i)\}_n$, then we have

$$\begin{aligned} &\mu_{\tilde{F}(x)}(y) = Sup\{\mu_F(f) | y = f(x)\} \\ &\Rightarrow \mu_{\tilde{F}(x)}(y) = \alpha_i \quad iff \quad y = f_i(x) \\ &\Rightarrow \tilde{F}(x) = \{(f_i(x), \alpha_i)\}_n \end{aligned} \tag{3.5}$$

Remark

Given two fuzzy set of mappings $F, G : X \to \tilde{P}(X)$ such that $\mu_F, \mu_G : \Re^X \to I$.

If F = G then the fuzzy functions $\tilde{F}, \tilde{G}$ which are constructed by F and G respectively are equal, i.e.

$$\mu_{\tilde{F}(x)}, \mu_{\tilde{G}(x)} : X \to I$$

$$\mu_{\tilde{F}(x)}(y) = \mu_{\tilde{G}(x)}(y) \quad \forall y \in X.$$

4. Fuzzy Functional

In this section, the concept of functional of ordinary function will be extended to a real fuzzy mapping.

Definition 3.5. Given a fuzzy function $F : X \to \tilde{P}(X)$ with $\mu_F : X \to I$ and a function $T : X \to Y$. Then there exists a fuzzy function $\tilde{T}(F) : Y \to \tilde{P}(Y)$ with $\mu_{\tilde{T}(F)} : Y \to I$ such that

$$\forall y \in Y, \ \mu_{\tilde{T}(F)}(y) = Sup\{\mu_F(x) | \forall x \in X, y = T(x)\} \tag{4.1}$$

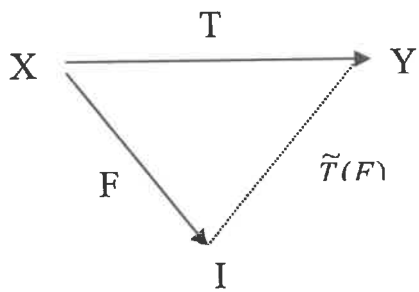

Fig.1 Fuzzy Function $\tilde{T}(F)$.

Definition 4.2. Given a fuzzy set of mappings $F : X \to \widetilde{P}(X)$ with $\mu_F : \Re^X \to I$ and a functional $\mathsf{F} : \Re^X \to \Re$. Then we can construct a fuzzy functional $\mathsf{F}^* : \widetilde{\Re}^X \to \widetilde{\Re}$ such that

$$\mathsf{F}^*(\widetilde{F}) = \widetilde{\mathsf{F}}(F)$$

Therefore, $\forall\ y \in \Re$

$$\mu_{\mathsf{F}^*(\widetilde{F})}(y) = \mu_{\widetilde{\mathsf{F}}(F)}(y) = Sup\{\mu_F(f) \mid f \in \Re^X, y = \mathsf{F}(f)\} \quad (4.2)$$

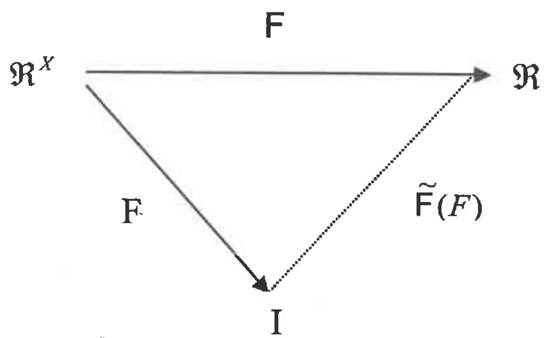

Fig.2 Fuzzy functional.

Example 3.7 let G be the set of all integrable functions. The integration $\int_X : G \subseteq \Re^X \to \Re$, can be considered as a functional where $\int_X f \in \Re$.

Then the fuzzy integral $\int_X^* : \widetilde{\Re}^X \to \widetilde{P}(\Re)$ can be defined according to (3.6) by:

Given a fuzzy mapping $\widetilde{F} : X \to \widetilde{P}(X)$, then $\exists F : X \to \widetilde{P}(X)$ with $\mu_F : G \subseteq \Re^X \to I$ such that

$$\mu_{\int_X^* \widetilde{F}}(y) = \mu_{\int_X F}(y) = Sup\{\mu_F(f) \mid f \in \Re^X, y = \int_X f\} \quad (4.3)$$

Example 3.8

We can define the differentiation of a fuzzy mapping at $x = x_\circ$ using equation (4.2) by:

Let P be the set of all differentiable functions $f : X \to \Re$ at $x = x_\circ$. let $D_{X_\circ} : P \subseteq \Re^X \to \Re$, where $D_{X_\circ}(f) = f'(x_\circ)$.

Then there exists a fuzzy derivative $D^*_{X_\circ} : \widetilde{\Re}^X \to \widetilde{\Re}$ such that at $x = x_\circ$

$$\mu_{D^*_X \widetilde{F}(x_\circ)}(y) = \mu_{\widetilde{D}_X F}(y) = Sup\{\mu_F(f) | f \in \Re^X, y = f'(x_\circ)\} \qquad (4.4)$$

Definition 3.9. Given a finite fuzzy mapping $F = \{(f_i, \alpha_i)\}_n$, and a functional $\mathsf{F} : \Re^X \to \Re$, then a fuzzy functional in this case, can be defined by

$$\mu_{\mathsf{F}^*(\widetilde{F})}(y) = \mu_{\widetilde{\mathsf{F}}(F)}(y) = Sup\{\alpha_i | \forall i = 1, \cdots, n, y = \mathsf{F}(f_i)\} \qquad (4.5)$$

5. Main Result (Fuzzy Operator)

In section 4, we considered a fuzzy mapping F such that $F : X \to \widetilde{P}(X)$ with $\mu_F : \Re^X \to I$. The functional of F over X was defined as a fuzzy set $\mathsf{F}^*(\widetilde{F})$. In this section, we shall deal with the operator of fuzzy function F, which will denoted by $\mathsf{L}^*(\widetilde{F})$.

Definition 5.1. Given a fuzzy function $F : X \to \widetilde{P}(X)$ with $\mu_F : \Re^X \to I$ and an operator $\mathsf{L} : \Re^X \to \Re^X$. Then we can construct a fuzzy operator $\mathsf{L}^* : \widetilde{\Re}^X \to \widetilde{\Re}^X$ such that

$$\mathsf{L}^*(\widetilde{F}) = \overline{\widetilde{\mathsf{L}}(F)}$$

Therefore, $\forall y \in \Re^X$

$$\mu_{\mathsf{L}^*(\widetilde{F})}(y) = \mu_{\overline{\widetilde{\mathsf{L}}(F)}}(y) = Sup\{\mu_{\widetilde{\mathsf{L}}(F)}(g) | \forall g \in \Re^X, y = g(x)\} \qquad (5.1)$$
$$= Sup\{Sup\{\mu_F(f) | \forall f, g \in \Re^X, g = \mathsf{L}(f), y = g(x)\}\}$$

When L is one-to-one operator. Then (5.1) will be

$$\mu_{\mathsf{L}^*(\widetilde{F})}(y) = Sup\{\mu_F(f) | \forall f \in \Re^X, y = \mathsf{L}(f)\} \qquad (5.2)$$

Example The Laplace transform L of a function $f : X \to Y$, is a one-to-one operator $\mathsf{L} : \Re^X \to \Re^X$ such that y = L (f) IFF

$$y(s) = \mathsf{L}(f)(s) = \int_0^\infty e^{-st} f(t) dt.$$

Then the fuzzy Laplace transform $\mathsf{L}^* : \widetilde{\Re}^X \to \widetilde{\Re}^X$ of a fuzzy function F can be constructed using (5.2) such that

$$\mathrm{L}^*(\tilde{F}) = \overline{\tilde{\mathrm{L}}(F)}$$

Therefore, $\forall x \in X,\ \forall y \in \Re^X$

$$\mu_{\mathrm{L}^*(\tilde{F})(s)}(y) = Sup\left\{ \mu_{\mathrm{F}}(f) \middle| \forall f \in \Re^X, y = \int_0^x e^{-st} f(t)dt \right\}$$

Definition 5.2. Given a finite fuzzy set of mappings $F = \{(f_i, \alpha_i)\}_n$, and an operator $\mathrm{L} : \Re^X \to \Re^X$. The fuzzy operator $\mathrm{L}^* : \tilde{\Re}^X \to \tilde{\Re}^X$ of F can be defined by

$\forall y \in \Re^X$

$$\mu_{\mathrm{L}^*(\tilde{F})}(y) = \mu_{\overline{\tilde{\mathrm{L}}(F)}}(y) = Sup\{\alpha_i | \forall i = 1, \cdots, n; y = \mathrm{L}(f_i)\} \quad (5.3)$$

If L is one-to-one then equation (3.28) will be:

$$\mu_{\mathrm{L}^*(\tilde{F})}(y) = \mu_{\overline{\tilde{\mathrm{L}}(F)}}(y) = \alpha_i \quad IFF \quad y = \mathrm{L}(f_i)$$

$$\Rightarrow \mathrm{L}^*(\tilde{F}) = \mathrm{L}^*\{(f_i, \alpha_i)\}_n = \{(\mathrm{L}(f_i), \alpha_i)\}_n \quad (5.4)$$

Remark Given a fuzzy mapping $\tilde{F}$ and $\tilde{G}$. Then we have

(*i*) $(\tilde{F} + \tilde{G})(x) = \tilde{F}(x) + \tilde{G}(x)$

(*ii*) $(\tilde{F} \cdot \tilde{G})(x) = \tilde{F}(x) \cdot \tilde{G}(x)$

(*iii*) $(c\tilde{F})(x) = c\tilde{F}(x)$

Lemma 5.1 Let F be a fuzzy mapping $\mu_F : X \to I$ and Let T and H be two operators such that $T : X \to Y,\ H : Y \to Z$, and H is one-to-one. Then we have

$$\tilde{H}\tilde{T}(F) = \overline{HT}(F) \quad (5.5)$$

proof:

$\forall z \in Z,$

$$\mu_{\tilde{H}\tilde{T}}(z) = Sup\{\mu_{\tilde{T}}(y) | \forall y \in Y, z = H(y)\}$$

$$= Sup\{Sup\{\mu_F(x) | \forall x \in X, y = T(x), \forall y \in Y, z = H(y)\}\}$$

Since H is one-to-one, then

$$\mu_{\tilde{H}\tilde{T}}(z) = Sup\{\mu_F(x) | \forall x \in X, z = H(T(x)) = (HT)(x)\}$$

$$= \mu_{\overline{HT}}(z).$$

Theorem 5.1 Let F be a fuzzy mapping $\mu_F : \Re^X \to I$ and I , L be two operators $I, \mathrm{L} : \Re^X \to \Re^X$ where L is one-to-one. Then there exist a fuzzy operators $I^*, \mathrm{L}^* : \widetilde{\Re}^X \to \widetilde{\Re}^X$ such that

$$\mathrm{L}^* I^*(\widetilde{F}) = (\mathrm{L}I)^*(\widetilde{F}) \tag{5.6}$$

Proof:

$$\mathrm{L}^* I^*(\widetilde{F}) = \mathrm{L}^*\left(\widetilde{I}(F)\right) = \widetilde{\mathrm{L}}\left(\overline{\widetilde{I}(F)}\right) = (\mathrm{L}I)^*(\widetilde{F})$$

By Lemma 5.1, we have

$$\widetilde{\mathrm{L}\widetilde{I}}(F) = \overline{\mathrm{L}I}(F)$$

$$\mathrm{L}^* I^*(\widetilde{F}) = \overline{\widetilde{\mathrm{L}\widetilde{I}}(F)} = \overline{\overline{\mathrm{L}I}(F)} = (\mathrm{L}I)^*(\widetilde{F}).$$

Example.

Let

$$g = I(f) \quad IFF \quad g(x) = \int_0^x f \quad and \quad g = \mathrm{L}(f) \quad IFF \quad g(s) = \int_0^\infty e^{-sx} f(x)dx.$$

Therefore

$$\mu_{\mathrm{L}^*\left(\int^* \widetilde{F}\right)(s)}(y) = \mu_{\left(\mathrm{L} \int_0^x \widetilde{F}\right)^*(s)}(y)$$

$$= Sup\left\{\mu_F(f) \middle| \forall f \in \Re^X ; y = \left(\mathrm{L} \int_0^x f\right)(s)\right\}$$

$$= Sup\left\{\mu_F(f) \middle| \forall f \in \Re^X ; y = \frac{1}{s}\mathrm{L}(f)(s)\right\}$$

Example 3.14.

Let $F = \{(f_1, \alpha_1), (f_2, \alpha_2), (f_3, \alpha_3)\}$

where $f_1(x) = e^x$, $f_2(x) = \sin x$, $f_3(x) = x$ and $\alpha_i = 0.4, 1, 0.1$, and we want to find the Laplace transform of F. Using (3.29),

$$\mathrm{L}^*(\widetilde{F}) == \{(\mathrm{L}(f_i), \alpha_i)\}_3 = \{(F_1, \alpha_1), (F_2, \alpha_2), (F_3, \alpha_3)\}$$

where

$$F_1(s) = \mathrm{L}(f_1) = \mathrm{L}(e^x) = \frac{1}{s-1}$$

$$F_2(s) = \mathrm{L}(f_2) = \mathrm{L}(\sin x) = \frac{1}{s^2+1}$$

$$F_3(s) = \mathrm{L}(f_3) = \mathrm{L}(x) = \frac{1}{s^2}$$

Theorem 5.2 ***Let*** $\widetilde{F}$ ***and*** $\widetilde{G}$ ***be real fuzzy mappings from X to the set*** $\widetilde{P}(X)$ ***such that*** $\widetilde{F} = \{(f_i, \alpha_i)\}_n, \widetilde{G} = \{(g_i, \beta_i)\}_n$. ***Then***

(i) $\mathrm{L}^*(\widetilde{F}+\widetilde{G}) = \mathrm{L}^*(\widetilde{F}) + \mathrm{L}^*(\widetilde{G})$ *or* $L^*(F(x)+G(x)) = L^*(F(x)) + L^*(G(x))$

(ii) $\mathrm{L}^*\left(\int_0^x \widetilde{F}(x-t)\widetilde{G}(t)dt\right) = \mathrm{L}^*(\widetilde{F})(s) \cdot \mathrm{L}^*(\widetilde{G})(s)$

Proof:

(i)

$$(\widetilde{F}+\widetilde{G})(x) = \widetilde{F}(x) + \widetilde{G}(x) = \{(f_i(x) + g_j(x)), \gamma_{ij}\}$$

where $\gamma_{ij} = \underset{i,j}{Max}\{Min(\alpha_i, \beta_j)\}$.

Then

$$\begin{aligned} L^*(\widetilde{F}(x)+\widetilde{G}(x)) &= \{(L(f_i(x)+g_j(x)), \gamma_{ij})\} \\ &= \{(L(f_i(x)) + L(g_j(x)), \gamma_{ij})\} \\ &= \{(L(f_i(x)), \alpha_i) + (L(g_j(x)), \beta_j)\} \\ &= L^*(\widetilde{F}(x)) + L^*(\widetilde{G}(x)) \end{aligned}$$

(ii)

$$\begin{aligned} L^*\left(\int_0^x \widetilde{F}(x-t)\widetilde{G}(t)dt\right) &= L^*\int_0^x (\{f_i(x-t), \alpha_i\} \cdot \{g_j(x), \beta_j\})dt \\ &= L^*\int_0^x \{f_i(x-t) \cdot g_j(x), \delta_{ij}\}dt \\ &= \left\{L\int_0^x (f_i(x-t) \cdot g_j(t))dt, \delta_{ij}\right\} \\ &= \{L(f_i(x)) \cdot L(g_j(t)), \delta_{ij}\} \\ &= \{\mathrm{L}(f_i)(s), \alpha_i\} \cdot \{\mathrm{L}(g_j)(s), \beta_j\} \\ &= \mathrm{L}^*(\widetilde{F})(s) \cdot \mathrm{L}^*(\widetilde{G})(s) \end{aligned}$$

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