SOLUTION OF LINEAR FUZZY INTEGRAL EQUATIONS

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Abstract

In this paper, we prove the existence of the solution of fuzzy integral equations of Volterra type by using the iteration of fuzzy operator. Also we find the exact solution of fuzzy linear integral equations by using convolution of fuzzy Laplace transformation and by using the method of successive approximation.

1. Introduction

Park et al[9] consider the existence of solutions of fuzzy integral equations in Banach space, and Subrahmanian and Sudarsanam[13] proved the existence of solutions of fuzzy functional equations. Jong Y.Park et al.[11] study the approximate solutions of the fuzzy functional integral equations.

In this work, we prove the existence of solution of the fuzzy integral equations by extending the fundamental theorem for ordinary integral equations. Also we study fuzzy Laplace transform and we apply it to find the exact solution of fuzzy integral equations.

$$\tilde{G}(x) = \tilde{F}(x) + \lambda \int_a^x \tilde{k}(x,t)\tilde{G}(t)dt$$

We assume that the fuzzy functions $\tilde{G}$, $\tilde{F}$, *and* $\tilde{k}$ are finite fuzzy numbers.

2. Definitions And preliminaries

(i) Fuzzy numbers are a convex unique normal with bounded support subsets of real numbers. In our work we will use fuzzy numbers with a fixed finite level set. By using the extension principle we extended the ordinary addition, subtraction, multiplication and division to operation on fuzzy numbers, also we extended the order relation to fuzzy numbers as well we define the distance between two fuzzy numbers as the supremum of the Hausdorff distance between two cuts. In [2] the set of fuzzy numbers is a complete ordered metric space. see[1]

(ii) Fuzzy linear equation on fuzzy real numbers can be solved by using finite level set of real numbers[1] as the fuzzy equation $\widetilde{a}\widetilde{x}+\widetilde{b}=\widetilde{x}$ where a $\widetilde{a},\widetilde{b}$ are level fuzzy numbers, which implies that the solution if it exists is[1]

$$\widetilde{x}=\left\{\left(\frac{b_i}{1-a_i},\alpha_i\right)\right\}_n \quad (2.1)$$

(iii) **Fuzzy function on fuzzy real numbers**

Definition 2.1 A fuzzy mapping $\widetilde{F}$ is a mapping from X to the set of non-empty fuzzy sets on Y, namely $\widetilde{P}(Y)$, In other words, to each $x \in X$, corresponds a fuzzy set $\widetilde{F}(x)$ defined on Y, whose membership function is $\mu_{\widetilde{F}(x)}$ and $\mu_{\widetilde{F}(x)} : Y \to I$.

When Y = R then the fuzzy mapping is fuzzy real mapping $\widetilde{F} : X \to \widetilde{R}$ with a membership function $\mu_{\widetilde{F}(x)} : R \to I$.

With each fuzzy function $\widetilde{F} : X \to \widetilde{P}(R)$, there is a function $F : \Re^X \to I$ such that

$$\mu_F(f) = Inf\{\mu_{\widetilde{F}(x)}(f(x)) | x \in X\},\ \forall f \in R^X,$$

And

$$\forall y \in R, \mu_{\widetilde{F}(x)}(y) = Sup\{\mu_F(f) | f \in R^X, y = f(x)\},$$

usually we take a subset of R^X such that for each x∈X, y∈R there exists a unique f in this subset of R^X, which implies that $\forall f$, $\mu_F(f) = \mu_{\widetilde{F}(X)}(f(x))$ for each $x \in X$, and for each $y \in R$, $\mu_{\widetilde{F}(X)}(y) = \mu_F(f)$ where y = f(x).

Let $\widetilde{F} : X \to \widetilde{R}$ then each $x \in X$, $\widetilde{F}(x)$ is a fuzzy real number by using the α-cut principle, then for each $x \in X$, for each α∈I

$$\left(\widetilde{F}(x)\right)_\alpha = \left[\widetilde{F}_\alpha^-(x), \widetilde{F}_\alpha^+(x)\right] = \widetilde{F}_\alpha(x),$$

then $\widetilde{F}_\alpha : X \to P(R)$, for each $x \in X$, $\widetilde{F}_\alpha(x)$ is closed interval

$$\left[\left(\widetilde{F}_\alpha(x)\right)^-, \left(\widetilde{F}_\alpha(x)\right)^+\right] = \left(\widetilde{F}(x)\right)_\alpha.$$

Moreover there exists two function $\widetilde{F}_\alpha^-, \widetilde{F}_\alpha^+ : X \to R$ such that

$$\widetilde{F}_\alpha(x) = \left[\widetilde{F}_\alpha^-(x), \widetilde{F}_\alpha^+(x)\right] \; or \; \widetilde{F}_\alpha = \left[\widetilde{F}_\alpha^-, \widetilde{F}_\alpha^+\right]$$

let $F = \{(f_i, \alpha_i)\}_n$ be a finite bunch of functions with $\mu_{\widetilde{F}}(f_i) = \alpha_i \; for \; i = 1,2,\dots,n$. Then for each $x \in X$, $\widetilde{F}(x) = \{(f_i(x), \alpha_i)\}_n$ where $\mu_{\widetilde{F}(x)}(f_i(x)) = \alpha_i \; for \; i = 1,2,\dots,n$. Therefore $\widetilde{F}(x)$ is a finite level fuzzy number.[3]

3-Fuzzy operators

Definition 3.1. Let $\mathrm{L} : \Re^X \to \Re^X$ be an operator on the set R^X (set of all functions from X to R). Then $\mathrm{L}^* : \widetilde{\Re}^X \to \widetilde{\Re}^X$ is a fuzzy operator defined as $\mathrm{L}^*(\widetilde{F}) = \overline{\widetilde{\mathrm{L}}(F)}$ for all fuzzy function $\widetilde{F} \in \widetilde{R}^X$.

From this definition $\mathrm{L}^*(\widetilde{F}) = \widetilde{G} \; where \; G = \widetilde{L}(F)$ for all $\widetilde{F} \in \widetilde{R}^X$, we know that $\mu_{\widetilde{G}(x)}(y) = G(f) \;\; where \;\; y = f(x)$. Therefore $\mathrm{L}^*(\widetilde{F}) = \widetilde{\mathrm{L}}(\mathrm{F})$ such that

$$\mu_{\mathrm{L}^*(F)(x)}(y) = \mu_{\widetilde{\mathrm{L}}(F).}(f) = \mu_{\mathrm{L}^*(F).}(f) = Sup\{\mu_{\mathrm{F}}(f) \mid g \in \Re^X, \mathrm{L}(g) = f, y = f(x)\}.$$

Then we write

$$\forall g \in R^X, \; \mu_{L^*(\widetilde{F})}(g) = \mu_{\widetilde{L}(F)}(g) = \sup\{\mu_F(f) \mid f \in R^X, L(f) = g\},$$

as an example when $L(f) = g$ where $g(x) = \int_a^x f$. Then

$\forall g \in R^X, X = [a,b]$

$$\mu_{L^*(\tilde{F})}(g) = \mu_{\tilde{L}(F)}(g) = \sup\left\{\mu_F(f) \middle| f \in R^X, g(x) = \int_a^x f\right\}$$

in case of Laplace transformation $L(f)$=g where $g(s) = \int_0^\infty e^{-st} f(t)dt$ we have

$$\mu_{L^*(\tilde{F})}(g) = \mu_{\tilde{L}(F)}(g) = \sup\left\{\mu_F(f) \middle| f \in R^X, g(s) = \int_0^\infty e^{-st} f(t)dt\right\}$$

in case when $F = \{(f_i, \alpha_i)\}_n$, then for all $g \in R^X$, $X = [a, \infty)$

$$\mu_{L^*(\tilde{F})}(g) = \mu_{\tilde{L}(F)}(g) = \sup\left\{\alpha_i \middle| f \in R^X, g(s) = \int_0^\infty e^{-st} f_i(t)dt\right\}$$

4. Existence Theorem For A solution Of Linear Fuzzy Integral Equations

In this section, we quote basic definitions [2,10,11] and theorems proved in [2,5,11], which will be needed in the proof of existence of solution of linear fuzzy integral equation

$$\tilde{G}(x) = \tilde{F}(x) + \lambda \int_a^x \tilde{k}(x,t)\tilde{G}(t)dt \qquad (4.1)$$

Let $\tilde{R}$ be the set of fuzzy real numbers, the distance between any two fuzzy numbers is given by

$$D(\tilde{a}, \tilde{b}) = \sup\{h(\tilde{a}_\alpha, \tilde{b}_\alpha) | \alpha \in I\} \qquad (4.2)$$

where h is the Haousdorff distance between closed intervals in R. we know that $(\tilde{R}, D)$ is a complete ordered metric space.

Let $\widetilde{F},\widetilde{G}: X \to \widetilde{R}$ be two fuzzy functions on X=[a,b] $\subseteq$ R. Let

$$D^*(\widetilde{F},\widetilde{G}) = Sup\{D(\widetilde{F}(X),\widetilde{G}(X))|x \in X\}$$

(4.3)

Therefore $(\widetilde{R}^X, D^*)$ is a complete metric space. Let $L^*: \widetilde{\Re}^X \to \widetilde{\Re}^X$ be a fuzzy operator we know that by a fuzzy fixed point theorem[5] that if

$$D^*(L^*(\widetilde{F}),L^*(\widetilde{G})) \le rD^*(\widetilde{F},\widetilde{G}) \quad where \quad 0 \le r < 1$$

for all $\widetilde{F},\widetilde{G} \in \widetilde{R}^X$, then there exists a unique $\widetilde{U} \in \widetilde{R}^X$ such that $L^*(\widetilde{U}) = \widetilde{U}$.

Now we let

$$L^*(\widetilde{G})x = \widetilde{F}(x) + \lambda \int_a^x \widetilde{k}(x,t)\widetilde{G}(t)dt \tag{4.4}$$

and $\widetilde{G}_{n+1} = L^*(\widetilde{G}_n)$ is a sequence in the complete metric space $\widetilde{R}^X$; has a fixed point with less condition than in the contraction operator in the fixed point theorem.

Definition 4.1. A fuzzy mapping $\widetilde{F}$: X → F($\Re$) is called levelwise continuous at $t_\circ \in X$ if the mapping $\widetilde{F}_\alpha$ is continuous at $t = t_\circ$ with respect to the Hausdorff metric D on F($\Re$) for all α∈]0,1]. As a special case when X=[a,b]⊆ $\Re$, this definition can be generalized to [a,b] x [a,b] as follows:

Definition 4.2. A fuzzy mapping $\widetilde{f}$: X x X → F($\Re$) is called levelwise continuous at point $(x_\circ, t_\circ) \in X \times X$ provided, for any fixed α∈[0,1] and arbitrary ε >0, there exists δ(ε,α)>0 such that

$$D([\widetilde{f}(x,t)]_\alpha, [\widetilde{f}(x_\circ,t_\circ)]_\alpha) < \varepsilon$$

whenever

$$|t - t_\circ| < \delta \ , \ |x - x_\circ| < \delta \quad \text{for all x , t} \in \text{X.}$$

Definition 4.3. Let $\widetilde{F} : X \to F(\Re)$, the integral of $\widetilde{F}$ over X =[a,b] denoted by $\int_X \widetilde{F}(t)dt$ is defined levelwise by the equation

$$\left[\int_X^* \widetilde{F}(t)dt\right]_\alpha = \int_X \widetilde{F}_\alpha(t)dt \quad for \quad all \quad 0<\alpha\leq 1 \tag{4.5}$$
$$=\left[\int_X \widetilde{F}_\alpha^-(t)dt, \int_X \widetilde{F}_\alpha^+(t)dt\right]$$

Theorem 4.1. If $\widetilde{F} : X \to F(\Re)$ levelwise continuous and Supp($\widetilde{F}$) is bounded, then F is integrable (Riemann integrable).

Theorem 4.2. Let F,G :X $\to$ F($\Re$) be integrable and $\lambda \in \Re$. Then

(i) $\int_X (F(t)+G(t))dt = \int_X F(t)dt + \int_X G(t)dt$,

(ii) $\int_X \lambda F(t)dt = \lambda \int_X F(t)dt$,

(iii) $D^*(F,G)$ is integrable,

(iv) $D^*\left(\int_X F(t)dt, \int_X G(t)dt\right) \leq \int_X D^*(F,G)(t)dt$

Now, we state and prove our resulting fundamental theorem which is the generalization of the ordinary existence theorem for a solution of integral equations. It is enough to show that the operator

$$T(\widetilde{u})(x) = \widetilde{f}(x) + \int_0^x \widetilde{k}(x,t)\widetilde{u}(t)dt \tag{4.6}$$

is contractive.

Theorem4.3. " Existence Theorem For A Solution Of Linear Fuzzy Integral Equations Of Volterra Type"

Consider the linear fuzzy integral equation of Volterra type

$$\tilde{u}(x) = \tilde{f}(x) + \int_0^x \tilde{k}(x,t)\tilde{u}(t)dt \qquad (4.7)$$

Assume that $T(\tilde{u})(x) = \tilde{u}(x)$

(i) The mapping $\tilde{f}$:X $\rightarrow$ F($\Re$) is levelwise continuous and Supp($\tilde{f}$) is bounded.

(ii) The mapping $\tilde{k}$: X x X$\rightarrow$ F($\Re$) is levelwise continuous.

(iii) The fuzzy mapping $\tilde{k}(x,t)$ has un upper bound such that

$$\forall \alpha \in]0,1] \quad , \quad \tilde{k}_\alpha(x,t) \leq M$$

where

$$0 \leq M < \frac{1}{|x-a|} \quad , \quad \forall x \in X$$

Then T is contractive.

Proof:

$$D^*(T(\tilde{u}),T(\tilde{v})) = \underset{x\in X}{Sup}\ \underset{\alpha\in]0,1]}{Sup}\ D\left(\{\tilde{f}(x)\}_\alpha + \left\{\int_a^x \tilde{k}(x,t)\tilde{u}(t)dt\right\}_\alpha , \{\tilde{f}(x)\}_\alpha + \left\{\int_a^x \tilde{k}(x,t)\tilde{v}(t)dt\right\}_\alpha\right)$$

From the result of theorem 5.6[15], we have

$$D^*\left(T(\tilde{u}),T(\tilde{v})\right)=\underset{x\in X}{Sup}\ \underset{\alpha\in]0,1]}{Sup}\ D\left(\left\{\int_a^x \tilde{k}(x,t)\tilde{u}(t)dt\right\}_\alpha,\left\{\int_a^x \tilde{k}(x,t)\tilde{v}(t)dt\right\}_\alpha\right)$$

$$=\underset{x\in X}{Sup}\ \underset{\alpha\in]0,1]}{Sup}\ Max\left\{\left|\int_a^x k_\alpha^-(x,t)u_\alpha^-(t)dt-\int_a^x k_\alpha^-(x,t)v_\alpha^-(t)dt\right|,\left|\int_a^x k_\alpha^+(x,t)u_\alpha^+(t)dt-\int_a^x k_\alpha^+(x,t)v_\alpha^+(t)dt\right|\right\}$$

$$=\underset{x\in X}{Sup}\ \underset{\alpha\in]0,1]}{Sup}\ Max\left\{\left|\int_a^x k_\alpha^-(x,t)\left(u_\alpha^-(t)-v_\alpha^-(t)\right)dt\right|,\left|\int_a^x k_\alpha^+(x,t)\left(u_\alpha^+(t)-v_\alpha^+(t)\right)dt\right|\right\}$$

$$\leq\underset{x\in X}{Sup}\ \underset{\alpha\in]0,1]}{Sup}\ Max\left\{\left|\int_a^x k_\alpha^-(x,t)\left|u_\alpha^-(t)-v_\alpha^-(t)\right|dt\right|,\left|\int_a^x k_\alpha^+(x,t)\left|u_\alpha^+(t)-v_\alpha^+(t)\right|dt\right|\right\}$$

$$\leq\underset{x\in X}{Sup}\ \underset{\alpha\in]0,1]}{Sup}\int_a^x \tilde{k}_\alpha(x,t)D\left(\tilde{u}_\alpha(t),\tilde{v}_\alpha(t)\right)dt$$

$$\leq\underset{x\in X}{Sup}\ \underset{\alpha\in]0,1]}{Sup}\int_a^x MD\left(\tilde{u}_\alpha(t),\tilde{v}_\alpha(t)\right)dt$$

$$=M\ \underset{x\in X}{Sup}\int_a^x M\tilde{D}\left(\tilde{u}(t),\tilde{v}(t)\right)dt$$

$$\leq M\int_a^x MD^*\left(\tilde{u}(t),\tilde{v}(t)\right)dt=M|x-a|D^*\left(\tilde{u},\tilde{v}\right).$$

Main Result (Solutions Of Fuzzy Integral Equations)

[1] By Fuzzy Laplace Transform Method

Consider the fuzzy integral equation with fuzzy difference kernel

$$\tilde{u}(x) = \tilde{f}(x) + \int_0^x \tilde{k}(x-t)\tilde{u}(t)dt$$

(4.8)

where

$$\tilde{u}(x) = \{(u_i(x), \alpha_i)\}_n, \tilde{f}(x) = \{(f_i(x), \alpha_i)\}_n, \tilde{k}(x-t) = \{(k_i(x-t), \alpha_i)\}_n$$

Taking fuzzy Laplace transform to both side of (4.8), we get

$$L^*(\tilde{u}(x)) = L^*(\tilde{f}(x)) + L^*\left(\int_0^x \tilde{k}(x-t)\tilde{u}(t)dt\right) \quad (4.9)$$

By convolution theorem , equation (4.9) will be

$$\mathrm{L}^*(\tilde{u})(s) = \mathrm{L}^*(\tilde{f})(s) + \mathrm{L}^*(\tilde{k})(s)\mathrm{L}^*(\tilde{u})(s) \quad (4.10)$$

$$\Rightarrow L^*\{(u_i(x), \alpha_i)\}_n = L^*\{(f_i(x), \alpha_i)\}_n + L^*\{(k_i(x), \alpha_i)\}_n L^*\{(u_i(x), \alpha_i)\}_n \quad (4.11)$$

$$\Rightarrow \{(L(u_i(x)), \alpha_i)\}_n = \{(L(f_i(x)), \alpha_i)\}_n + \{(L(k_i(x)), \alpha_i)\}_n \{(L(u_i(x)), \alpha_i)\}_n$$

$$\Rightarrow \{(L(u_i(x)), \alpha_i)\}_n = \{(L(f_i(x)), \alpha_i)\}_n + \{(L(k_i(x))L(u_i(x)), \alpha_i)\}_n$$

$$\{(L(u_i(x)), \alpha_i)\}_n = \{(L(f_i(x)) + L(k_i(x))L(u_i(x)), \alpha_i)\}_n$$

Hence,

$$\forall \alpha \in]0,1],$$

$$L(u_i(x)) = L(f_i(x)) + L(k_i(x))L(u_i(x)), \quad \forall i = 1,2,\dots,n$$

$$\mathrm{L}(u_i)(s) = \mathrm{L}(f_i)(s) + \mathrm{L}(k_i)(\mathrm{s})\mathrm{L}(u_i)(s)$$

$$\Rightarrow \mathrm{L}(u_i)(s) = \frac{\mathrm{L}(f_i)(s)}{1 - \mathrm{L}(k_i)(\mathrm{s})}, \mathrm{L}(k_i)(\mathrm{s}) \neq 1$$

$$\Rightarrow u_i(x) = \mathrm{L}^{-1}(\mathrm{L}(u_i)(s)) = \mathrm{L}^{-1}\left(\frac{\mathrm{L}(f_i)(s)}{1 - \mathrm{L}(k_i)(\mathrm{s})}\right)$$

So the solution of (4.8) is

$$\tilde{u}(x) = \{(u_i(x), \alpha_i)\}_n = \left\{ \left(L^{-1}\left(\frac{L(f_i)(s)}{1 - L(k_i)(s)} \right), \alpha \right) \right\}_{in} \quad (4.12)$$

Example 4.1

Let $\tilde{u}(x) = \tilde{f}(x) + \int_0^x \tilde{k}(x-t)\tilde{u}(t)dt$

Where

$$\tilde{f}(x) = \{(f_1(x), 0.3), (f_2(x), 1.0), (f_3(x), 0.5)\}$$
$$\tilde{k}(x-t) = \{(k_1(x-t), 0.3), (k_2(x-t), 1.0), (k_3(x-t), 0.5)\}$$

and

$$f_1(x) = \frac{x}{2}, \quad f_2(x) = x, \quad f_3(x) = 5x$$
$$k_1(x-t) = -2(x-t), \quad k_2(x-t) = -(x-t), \quad k_3(x-t) = (x-t)$$

First we have

$$L(f_1)(s) = \frac{1}{2s^2}, \quad L(f_2)(s) = \frac{1}{s^2}, \quad L(f_3)(s) = \frac{5}{s^2},$$
$$L(k_1)(s) = \frac{-2}{s^2}, \quad L(k_2)(s) = \frac{-1}{s^2}, \quad L(k_3)(s) = \frac{1}{s^2}$$

Now, from equation (4.12), we have

$$u_i(x) = L^{-1}\left(\frac{L(f_i)(s)}{1 - L(k_i)(s)} \right) \quad \forall \alpha_i \in]0,1]$$

$$\therefore u_1(x) = L^{-1}\left(\frac{1/2s^2}{1+2/s^2}\right) = \frac{1}{2\sqrt{2}}\sin 2x$$

$$u_2(x) = L^{-1}\left(\frac{1/s^2}{1+1/s^2}\right) = \sin x$$

$$u_3(x) = L^{-1}\left(\frac{5/s^2}{1-1/s^2}\right) = 5\sinh x$$

$$\therefore u(x) = \left\{\left(\frac{1}{2\sqrt{2}}\sin 2x, 0.3\right), (\sin x, 1.0), (5\sinh x, 0.5),\right\}.$$

[2] By Method of Successive Approximation

We illustrate here how the method of successive approximation can be used to solve the fuzzy integral equation

$$\tilde{u}(x) = \tilde{f}(x) + \int_0^x \tilde{k}(x,t)\tilde{u}(t)dt \tag{4.13}$$

where $\tilde{u}(x) = \{(u_i(x),\alpha_i)\}_n, \tilde{f}(x) = \{(f_i(x),\alpha_i)\}_n, \tilde{k}(x,t) = \{(k_i(x,t),\alpha_i)\}_n$

then $$\{(u_i(x),\alpha_i)\}_n = \{(f_i(x),\alpha_i)\}_n + \int_0^x \{(k_i(x,t),\alpha_i)\}_n \{(u_i(t),\alpha_i)\}_n dt \tag{4.14}$$

Now , equation (4.14), can be written as:

$$\{(u_i(x),\alpha_i)\}_n = \{(f_i(x),\alpha_i)\}_n + \left\{\left(\int_0^x k_i(x,t)u_i(t)dt, \alpha_i\right)\right\}_n \tag{4.15}$$

and by equation (2.1),

$$\{(u_i(x),\alpha_i)\}_n = \left\{\left(f_i(x) + \int_0^x k_i(x,t)u_i(t)dt, \alpha_i\right)\right\}_n \tag{4.16}$$

which implies that for each $\forall \alpha_i \in]0,1]$

$$u_i(x) = f_i(x) + \int_0^x k_i(x,t)u_i(t)dt \quad \forall i = 1,2,\dots,n \tag{4.17}$$

Now, we can apply the method of successive approximation to equation(4.17), so we will get

$$u_i^{(m+1)}(x) = f_i(x) + \int_0^x k_i(x,t) u_i^{(m)}(t)dt \quad \forall i = 1,2,\ldots,n \tag{4.18}$$

$$\Rightarrow \left\{\left(u_i^{(m+1)}(x), \alpha_i\right)\right\}_n = \left\{\left(f_i(x) + \int_0^x k_i(x,t) u_i^{(m)}(t)dt, \alpha_i \right)\right\}_n \tag{4.19}$$

Example 4.2

Consider the linear fuzzy integral equation

$$\tilde{u}(x) = \tilde{f}(x) + \int_0^x \tilde{k}(x,t)\tilde{u}(t)dt$$

Where

$$\tilde{f}(x) = \{(f_1(x), 0.4), (f_2(x), 1.0)\}$$
$$\tilde{k}(x-t) = \{(k_1(x,t), 0.4), (k_2(x,t), 1.0)\}$$

and

$$f_1(x) = x, \quad f_2(x) = 1(x) = 5x$$
$$k_1(x,t) = -(x-t), \qquad k_2(x,t) = 1$$

Applying equation (4.19), we get

at $\alpha = 0.4$,

$$u_1^{(m+1)}(x) = f_1(x) + \int_0^x k_1(x,t) u_1^{(m)}(t)dt$$

$$u_1^{(m+1)}(x) = x - \int_0^x (x-t) u_1^{(m)}(t)dt \tag{E.2}$$

we shall start with $u_1^{(0)}(t) = 0$ in the integral of (E.2) to obtain $u_1^{(1)}(t)$

$$\therefore u_1^{(1)}(x) = x - 0 = x$$

Now

$$u_1^{(2)}(x) = x - \int_0^x (x-t)t\,dt = x - \frac{x^3}{3!}$$

$$u_1^{(3)}(x) = x - \int_0^x (x-t)\left(t - \frac{t^3}{3!}\right)dt = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

If we continue this process, we obtain the m-th approximation $u_1^{(m)}(x)$ as

$$u_1^{(m)}(x) = \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^{(m)}\frac{x^{2m+1}}{(2m+1)!}\right]$$

Which is the m-th partial sum of the Maclaurin series of sinx. Hence the solution $u_1(x)$ to equation (E.1) is

$$u_1(x) = \lim_{m\to\infty} u_1^{(m)}(x) = \sin x.$$

Now, at α = 1.0, equation (4.19) reduced to

$$u_2^{(m+1)}(x) = f_2(x) + \int_0^x k_2(x,t)u_2^{(m)}(t)dt$$

$$u_2^{(m+1)}(x) = 1 - \int_0^x (1)u_2^{(m)}(t)dt \qquad \text{(E.3)}$$

Starting with $u_2^{(0)}(t) = 0$,we get

$$u_2^{(1)}(x) = 1 + 0 = 1$$

$$u_2^{(2)}(x) = 1 + \int_0^x (1)dt = 1 + x$$

$$u_1^{(3)}(x) = 1 + \int_0^x (1+t)dt = 1 + x + \frac{x^2}{2}$$

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$$u_1^{(m)}(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots + \frac{x^m}{m!}$$

$$\Rightarrow u_2(x) = \lim_{m\to\infty} u_2^{(m)}(x) = e^x$$

So the solution of the fuzzy integral equation (E.1) is :

$$u(x) = \{(\sin x, 0.4), (e^x, 1.0)\}.$$

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