SOME PROPERTIES AND ZERO-DIMENSIONALITY OF FUZZY METRIC SPACES

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Abstract:

The purpose of this paper is to introduce the concepts of closure ,limit points and boundary of set in fuzzy metric spaces. Also we introduce some topological properties and the concepts of zero-dimensional and small inductive dimension in fuzzy metric space, some important and interesting results are obtained .

Keywords: Fuzzy metric spaces, Limit pints, boundary ,zero-dimensional, small inductive dimension

1. Introduction:

In 1965, the concept of fuzzy set was introduced by Zadeh [18]. One of the most important problems in fuzzy topology is to obtain an appropriate concept of fuzzy metric space .This problem has been investigated by Many authors [3,4,7,12,13].They introduced the concept of fuzzy metric space in different ways. In particular George and Veeramani [7] have introduced and studied a notion of fuzzy metric space with the help of continuous t-norms, which constitutes a slight but appealing modification of the one due to Kramosil and Michalek [12] and defined a Hausdorff topology on this fuzzy metric space. Nathanson [5] introduced the concept of round metric space and gave the relationship between round metrics and equivalent metrics. Also Many authors have studied fixed theory in fuzzy metric spaces such as[1,2,9,10,11,16] .

The aim of this paper is to extend some concepts to fuzzy metric spaces such as limit point and boundary ,we obtain some results about them, we investigate two properties of non-Archimedean fuzzy metric space, also we prove that every fuzzy metric space is normal, Every separable fuzzy metric spaces is second countable, And we introduce the concept of zero – dimensionality some results about this concept are given.

2.Preliminaries

We begin with some definitions.

Definition:2.1[15] A binary operation *:[0,1]×[0,1] → [0,1] is a continuous triangular norm (shortly t-norm) if * satisfies the following conditions:

1.* is associative and commutative.

2. * is continuous.

3. a*1= a for all a∈ [0,1].

4. a*b ≤ c*d whenever a ≤ c and b ≤ d for all a,b,c,d∈ [0,1].

Example(2.2) The following are examples of t-norms:

(1). a *b = ab. (2) a * b = min {a, b}. (3) a*b = max {0,a+b−1}.

Definition2.3[7] A fuzzy metric space is an ordered triple (X,M, *) such that X is an arbitrary (non-empty) set, * is a continuous t-norm and M is a function defined on $X^2 \times]0,+\infty[$ with values in]0,1[satisfying the following conditions, for all x,y,z ∈ X and s,t > 0 :

(i) M(x,y,t) > 0,

(ii) M(x,y,t) =1 if and only if x = y,

(iii) M(x,y,t) = M(y,x,t),

(iv) M(x,y,t) *M(y,z,s) ≤ M(x,z,t+s),

(v) M(x,y,·):]0,+∞[→[0,1] is continuos.

Then M is called a fuzzy metric on X. The function M(x,y,t) denote the degree of nearness between x and y with respect to t, also condition (ii) is equivalent to M(x,x,t) = 1 for all x ∈ X and t > 0 , and M(x,y,t) < 1 for all x ≠ y and t > 0 .

Remark 2.4 [9] In fuzzy metric space X, M(x, y, .) is non-decreasing for all x, y ∈ X.

Example2.5 Let (X, d) be a metric space. Denote a * b = ab for all a, b ∈ [0, 1] and let M_d be a fuzzy set on $X^2 \times]0,\infty[$ defined as follows:

$$M_d(x, y, t) = \frac{kt^n}{kt^n + md(x, y)}$$

for all k, m, n∈R^+, x ,y∈ X , Then (X, M_d, *) is a fuzzy metric space.

Remark 2.6 Note the above example holds even with the t-norm a * b = min{a, b} and hence M is a fuzzy metric with respect to any continuous t-norm.

In above example by putting $k = m = n = 1$, we get

$$M_d (x, y, t) = \frac{t}{t + d(x, y)}$$

We call this fuzzy metric induced by a metric d the standard fuzzy metric.

Definition 2.7[7] Let (X,M, *)be a fuzzy metric space and let $r \in (0, 1), t > 0$ and $x \in X$. The set B(x, r, t) = {y $\in$X : M(x, y, t) > 1 − r} is called the open ball with center *x* and radius *r* with respect to *t*.

Theorem 2.8[7] Every open ball $B(x, r, t)$ is an open set.

George and Veeramani proved in [7] that every fuzzy metric space (X,M, *) on X generates a topology τ_M on X which has as a base the family of open sets of the form {B_M(x ,r ,t) : x $\in$X; 0< r <1 ,t > 0} ,they proved that (X,τ_M) is Hausdorff first countable topological space, where τ_M = {A $\subset$ X :for each x $\in$ X, there exist t > 0, r $\in$(0, 1) such that B(x, r, t) $\subset$ A}.Also if (X,τ) metric space, then the topology induced by d coincides with the topology τ_{M_d} induced by the fuzzy metric M_d.

Example2.9 Let X = N (where N is the set of neutral number) and we define a * b = max{0, a + b −1} for all a, b $\in$ [0, 1] and let M be a fuzzy set on $X^2 \times (0,\infty)$ defined as follows:

$$M(x,y,t) = \begin{cases} \dfrac{x}{y} & ,if \quad x \le y \\ \dfrac{y}{x} & ,if \quad y \le x \end{cases}$$

for all x, y $\in$X, and t > 0 then (X,M, *) is a fuzzy metric space. M induces on X the discrete topology , (in fact ,for x $\neq$ y we have M(x, y, t) $\le \dfrac{x}{(x+1)}$ .Now, if we choose r such that

$0 < r < \dfrac{x}{(x+1)}$,then y $\in$B(x, r, t) if and only if M(x, y, t) > 1 − r > $\dfrac{x}{(x+1)}$ and ,therefore, B(x, r, t) ={x}).

Theorem 2.10[7] Let (X,M,*) is a fuzzy metric space .Then τ_M is a Hausdorff topology and for each x $\in$X ,{B(x,$\frac{1}{n}$,$\frac{1}{n}$): n$\in$N} is a neighborhood base at x for the topologyτ_M.

Gregori and Romaguera proved in [10] that if (X,M,*) is a fuzzy metric space, then

{U_n: n$\in$N} is a base for a uniformity U on X compatible with τ_M, where

U_n = {(x; y)∈X ×X : $M(x,y,\frac{1}{n})$ >1 − $\frac{1}{n}$} for all n∈N. Therefore (X,τ_M) is a metrizable topological space. Also they proved that a topological space   is metrizable if and only if admits a compatible fuzzy metric.

Theorem2.11[8] Every separable fuzzy metric spaces is second countable.

Definition 2.12[8] Let (X,M, *) be a fuzzy metric space and let r ∈ (0, 1),t > 0 and x ∈ X. The set B[x, r, t] = {y ∈ X : M(x, y, t) ≥ 1 − r} is called the closed ball with center x and radius r with respect to t.

Theorem 2.13[8] Every closed ball $B[x, r, t]$ is a closed set.

Definition.2.14 A subset of fuzzy metric space is said to be clopen if and only if it is closed and open.

Theorem 2.15 [7] A sequence(x_n) in a fuzzy metric space (X,M, *) converges to x if and only if $M(x_n, x, t) \to 1$ as n→∞.

Example 2.16 Let X = **R**, the set of all real numbers.
For x; y ∈X; t ≥ 0 , define

$$M(x,y,t) = \begin{cases} \dfrac{t}{t+|x-y|} & ,if \quad t > 0 \\ 0 & ,if \quad t = 0 \end{cases}$$

Then M is a fuzzy metric on **R** . Let (s_n) be a sequence defined as
$s_n = \dfrac{1}{n}$,for n∈ N. Then $M(s_n,x,t) \to 1$ as n→∞ .

Definition 2.17 [8]. A sequence(x_n) in a fuzzy metric space (X,M, *) is a Cauchy sequence if and only if for each r∈(0, 1) and each t > 0 there exists n_0 ∈N such that $M(x_n, x_m, t)$ >1 − r for all n ,m ≥ n_0

Definition 2.18 [7] A fuzzy metric space in which every Cauchy sequence is convergent is called a complete fuzzy metric space.

Example 2.19. Let X = **R**⁺, with the metric d defined by d(x,y) = |x− y| ,and defined

$$M(x, y, t) = \frac{t}{t+d(x,y)}$$,for all x ,y ∈ X ,t > 0.Clearly (X,M, *) is a complete fuzzy metric spaces.

Theorem 2.20 [8] Let $(X,M, *)$ be a fuzzy metric space ,then for each metric d on X compatible with M ,the following hold;

1. A sequence(x_n) in X is Cauchy in $(X,M,*)$ if and only if it is Cauchy in (X, d).

2. $(X,M, *)$ is complete if and only if (X,d) is complete.

Lemma 2.21 Let $(X,M, *)$ be a fuzzy metric space. If $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ then $\lim_{n\to\infty} M(x_n, y_n, *) = M(x, y, *)$.

Definition 2.22 [17] Let $(X,M, *)$ be a fuzzy metric space, $x \in X$ and $A \subset X$. The distance between x and A is defined by
$M(A, x, t) = \sup\{M(y, x, t) : y \in A\}$ for all $t > 0$.

Definition 2.23 A fuzzy metric space $(X,M, *)$ is called non-Archimedean if
$M(x,z,t) \geq \min\{M(x,y,t), M(y,z,t)\}$,for all $x,y,z \in X$.

Clearly ,if $(X,M, *)$ is a non-Archimedean fuzzy metric ,and $*$ is a t. norm defined by
$x*y = \min\{x,y\}$,then it's fuzzy metric.

Proposition 2.24 Let d be a metric on X ,and M_d the corresponding standard fuzzy metric .Then d is non-Archimedean if and only if M_d is non-Archimedean.

Proof. Suppose d is non-Archimedean. Then
$M_d(x,z,t) =$
$$\frac{t}{t+d(x,z)} \geq \frac{t}{t+\max\{d(x,y),d(y,z)\}} = \min\{M_d(x,y,t).M_d(y,z,t)\}.$$
Conversely, $M_d(x,z,t) = \dfrac{t}{t+d(x,z)}$, and then $\dfrac{t+d(x,z)}{t} = \dfrac{1}{M_d(x,y,z)}$,this
implies that $d(x,z) = \dfrac{t}{M_d(x,y,z)} - t = t(\dfrac{1}{M_d(x,y,z)} - 1)$
$$\leq t(\frac{1}{\min\{M_d(x,y,t),M_d(y,z,t)\}} - 1) =$$
$\max\{d(x,y),d(y,z)\}$. ∎

3-Limits point , boundary and Some topological properties in fuzzy metric spaces

In the first of this section, we introduce the concept of limit point

Definition 3.1 Let $(X, M, ^*)$ be a fuzzy metric space ,and A sub set of X, we say that a point $x \in X$ is limit point of a set A if and only if $\forall$ r $(0 < r < 1$), t > 0 the open ball $B(x, r, t)) \cap A \backslash \{x\} \neq \varnothing$.

We denote the set of all limit point of A by the set d(A) or A' ,and a subset A in fuzzy metric space $(X, M, ^*)$ is dense in X if every point of X is a limit point of A. we prove the following theorem:

Theorem3.2 We say that $x \in d(A)$ if and only if there exist a sequence (a_n) convergence

to x, where $a_n \in A$, $a_n \neq x$.

Proof. Suppose $x \in d(A)$,then for t >0 ,0 < r<1,there is an element a_1 in A such that $a_1 \in B(x, r, t)$ i.e. $M(a_1, x, t) > 1 - r_1$, $a_1 \neq x$, let $0 < r_2 < r_1$, then there is an element a_2 in A such that $a_2 \in B(x, r_2, t)$ this implies that $M(a_2, x, t) > 1 - r_2 > 1 - r_1$, $a_2 \neq x$, again by this way we get the sequences (a_n), (r_n) in A are constructed such that $M(a_n, x, t) > 1 - r_n$, $r_n \to 0$, $a_n \neq x$, i.e. $a_n \to x$.

Conversely ,let(a_n) be a sequence convergence to x , $a_n \neq x$ and $a_n \in A$ in $(X, M, ^*)$,let U be any neighborhood of x then f or any r ,t > 0 ,0 < r < 1 there is

$B(x, r, t) = \{$ y: $M(x, y, t) > 1 - r\} \subset U$ this implies $y \in B$ and corresponding to r there exist a positive integer N such that $n \geq N$,and since $a_n \to x$ then $M(a_n, x, t) > 1 - \frac{1}{n}$ let

$r < \frac{1}{n}$ and clearly $M(a_n, x, t) \to 1$ as $n \to \infty$ then $B(a_n, \frac{1}{n}, t) \subset B(x, r, t)$ and

$M(a_n, x, t) > 1 - \frac{1}{n} > 1 - r$. Thus $a_n \in B(x, r, t)$ and hence $a_n \in U$. ∎

We give now the concept of boundary of set in fuzzy metric spaces:

Definition3.3 Let $(X, M, ^*)$ be a fuzzy metric space ,and we say that x is boundary point of A if there exist an open ball B(x,r,t) centered at x such that B(x,r,t) $\cap$ A≠∅ and

B(x,r,t) $\cap$ X\A ≠ ∅ for every 0 <r < 1. And the boundary of A is the set of all boundary points of A, denoted by ∂A.

Theorem.3.4 We say that $x \in \partial A$ if and only if there exist a sequence of limit points $(a'_n) \in X \backslash A$ convergent to x ,and a sequence $(a_n) \in A$ convergent to x.

proof Suppose $x \in \partial A$,then for any $0 < r < 1$ the ball $B(x, r, t)$ contains points out of both
A(i.e. the point a_r) ,then $M(x, a_r, t) > 1 - r$, and X\A(i.e. the point a'_r),then $M(x, a'_r, t) > 1 - r$ From theorem (3.2) assuming that $r = r_n$, and $r_n \to 0$ we obtain the sequences $(a_r) \in A$ and $(a'_r) \in X \backslash A$ such that $a_{r_n} \to x$ and $a'_{r_n} \to x$.

Conversely ,if $a_n \to x$, $(a_n) \in A$ and $a'_n \to x$ $(a'_n) \in X \backslash A$, then any ball $B(x,r,t)$ contains the points a_n and the points a'_n for all sufficiently large $n = n(r)$, therefore $x \in \partial A$. ∎

Proposition 3.5 Let $(X,M,^*)$be a fuzzy metric space, $A \subseteq X$,$T \subseteq X$,then $\partial_T (A \cap T) \subseteq \partial_X A$.

Proof. Let $x \in \partial_T (A \cap T)$. then for every r ,$t > 0$, $0 < r < 1$ there exist a points $y \in A \cap T$ with
$M(x, y, t) > 1 - r$ and a points $z \in T \backslash (A \cap T)$ such that $M(x, z, t) > 1 - r$. But point such as y are points of A, and points such as z are points of X\A, so x is a boundary point of A in $(X,M,^*)$. ∎

We prove the following theorem:

Theorem3.6 Every fuzzy metric space $(X,M,\ ^*)$ is normal.

Proof. Let $(X,M,^*)$ be a fuzzy metric space ,and F ,G be disjoint closed subsets of X . Let $x \in F$ then $x \in G^c$,since G^c is open ,there exist $t_x > 0$,and $0 < r_x < 1$.such that $B(x,r_x,t) \cap G = \varnothing$ for all $x \in F$. Similarly there exist $t_y > 0$,and $0 < r_y < 1$ such that $B(x,r_y,t) \cap F = \varnothing$, for all $y \in G$. Let $r = \min\{r_x, r_y\}$,and $t = \min\{\frac{t_x}{2}, \frac{t_y}{2}\}$ then for given $0 < r_0 < 1$ we can find r such that$(1-r_0) \ ^* (1-r_0) > 1-r$.

But $U = \bigcup_{x \in F} B(x, r_0, \frac{t}{2})$,and $V = \bigcup_{y \in G} B(y, r_0, \frac{t}{2})$ then U and V are open sets containing F and G respectively ,now we claim that $U \cap V = \varnothing$,let $z \in U \cap V$ then there exist $x \in F$ and $y \in G$ such that $z \in B(x, r_0, \frac{t}{2})$ .and $z \in B(y, r_0, \frac{t}{2})$,then

$M(x, y, t) \geq M(x, z, \frac{t}{2}) \ ^* M(z, y, \frac{t}{2}) > (1-r_0) \ ^* (1-r_0) > 1-r$. Hence $y \in B(x,r,t)$, but since
$r < r_x$, $t < t_x$,$B(x,r,t) \subset B(x,r_x,t_x)$ and, thus $B(x,r_x,t_x) \cap G \neq \varnothing$ which is contradiction . Hence X is normal . ∎

Definition 3.7 [10] A fuzzy metric space (X,M,*) is called precompact if for each $0 < r < 1$, and each $t > 0$, there is a finite subset A of X, such that $X = \bigcup_{a \in A} B(a, r, t)$ In this case, we say that M is a precompact fuzzy metric on X.

Theorem 3.8[10] A fuzzy metric space is precompact if and only if every sequence has a Cauchy subsequence.

Definition 3.9[10] A fuzzy metric space (X,M,*) is called compact if (X, τ_M) is a compact topological space.

Theorem 3.10 [10] A fuzzy metric space is compact if and only if it is precompact and complete.

Also ,we prove the following theorem:

Theorem 3.11 Every compact fuzzy metric space is separable.

Proof. Let (X,M, *) be the given compact fuzzy metric space. Let$0 < r<0$,t > 0 and $t > 0$. Since X is compact ,there exist x_1, x_2 , . . . ,x_n in X such that $X = \bigcup_{i=1}^{n} B(x_i, r, t)$. In particular, for each n $\in$ N, we can choose a finite subset A_n such that $X = \bigcup_{n \in A_n}^{n} B(x_i, r_n, \tfrac{1}{n})$
in which $r_n \to 0$ .Let $A = \bigcup_{n \in N} A_n$. Then A is countable. We claim that $X \subset \overline{A}$ Let x$\in$X. Then for each n $\in$ N, there exists $a_n \in A_n$ such that $x \in B(a_n, r_n, \tfrac{1}{n})$

Thus a_n converges to x. But since $a_n \in$ A for each $x \in \overline{A}$. Hence A is dense in X and thus X is separable. ∎

4- Zero-Dimensional

Recall that in the usual metric space (**R**,d) the open base consisting of open set of the form B(x,r) = (x–r, x+r).
For usual fuzzy metric space (X,M,*) ,where X = **R** ,and for example M is defined by M(x, y, t) = $1 - (\dfrac{1}{x \wedge y} - \dfrac{1}{x \vee y})$ for all x,y $\in$ **R** ,t > 0,and * is defined by

$x*y = \max\{0, x + y - 1\}$, $r < 1/x$, $x \neq 0$, then it is easy to verify that the open base is given by

$B(x,r,t) = (\dfrac{x}{1-rx}, \dfrac{x}{1+rx})$, $B(x,r,t)$ is an open interval or $\mathbf{R}$ whose diameter converges to zero as $r \to 0$.

Definition4.1 A fuzzy metric space $(X, M, *)$ is zero dimensional if and only if there exists a base for the open set consisting of clopen set.

Example4.2 A non-Archimedean fuzzy metric space $(X, M, *)$ is zero dimensional.

Exapme4.3 Let $(X, M, *)$ be fuzzy metric space, $X = \{x_1, x_2, \ldots, x_n\}$ and we let $r = \min\{M(x_i, x_j, t), t > 0, i \neq j\}$ then all of open balls $B(x_j, r, t)$ is clopen and these balls constitute a base for the open sets of $(X, M, *)$.

Now, we prove the following theorem:

Theorem4.4 Let (X, τ) be T_0 topological space then (x, τ) is strongly zero-dimensional and metrizable if and only if (x, τ_M) is non-Archimedean fuzzy metrizable.

Proof. Suppose (X, τ) is strongly zero-dimensional and metrizable, then from [6] it is non-Archimedean metrizable, and from proposition (2-24) it is non-Archimedean fuzzy metrizable.

Conversely, suppose $(X, M, *)$ is compatible non-Archimedean fuzzy metric in (X, τ) then the family $\{U_n : n \in \mathbf{N}\}$ where $U_n = \{(x, y) \in X \times X : M(x, y, \frac{1}{n}) > 1 - \frac{1}{n}\}$ is

abase for uniformly $\mathbf{U}$ on X which is compatible with τ_M, if $(x,y), (y,z) \in U_n$ then

$M(x, z, \frac{1}{n}) \geq \min\{M(x, y, \frac{1}{n}), M(y, z, \frac{1}{n})\} > 1 - \frac{1}{n}$, then $(x,z) \in U_n$, thus $U_n \circ U_n$

$\subset U_n$ i.e. U_n is transitive, therefore from [6] the Hausdorff topological space(X, τ) is strongly zero-dimensional and metrizable space. ∎

Theorem4.5. The only clopen sets in fuzzy metric space $(\mathbf{R}, M, *)$ are the sets $\mathbf{R}$ and $\varnothing$, therefore $(\mathbf{R}, M_d, *)$ is not zero-dimensional.

Proof. Let $(\mathbf{R}, M_d, *)$ be a fuzzy metric space, and A be an arbitrary open set in $\mathbf{R}$, and suppose $A \neq \varnothing$ and $A \neq \mathbf{R}$, we must show that A is not clopen set or equivalently that A has a boundary point .let (x_n), (y_n) be two sequences in $\mathbf{R}$.

First , since $A \neq \varnothing$,we my choose a point $x_0 \in A$, there exist ,$r,t > 0$,$0 < r < 1$ such that

$B(x_0,r,t) \subset A$, and since $A \neq \mathbf{R}$ we my choose, $y_0 \notin A$ .After x_n and y_n have been defined ,with $x_n \in A$ which implies that there exist ,$r,t > 0$,$0 < r < 1$ such that $B(x_n,r,t) \subset A$ and $y_n \notin A$ We want to define x_{n+1} and y_{n+1} .

Consider the midpoint $z_n = \dfrac{x_n + y_n}{2}$,if $z_n \in A$, there exist $r,t > 0$, $0 < r < 1$ such that

$B(z_n,r,t) \subset A$, then define $x_{n+1} = z_n$,$y_{n+1} = y_n$,and if $z_n \notin A$ then define $x_{n+1} = z_n$, $y_{n+1} = y_n$ so in any case we get $x_{n+1} \in A$ i.e. there exist $r,t > 0$, $0 < r < 1$ such that $B(x_{n+1},r,t) \subset A$, and $y_{n+1} \notin A$,with $M_d(x_{n+1}, \quad y_{n+1},t)$

$$= \frac{t}{t + |x_{n+1} - y_{n+1}|} = \frac{t}{t + \left|\dfrac{x_n + y_n}{2} - y_n\right|} = \frac{t}{t + \left|\dfrac{x_n - y_n}{2}\right|} \text{,then} \quad \text{by} \quad \text{induction}$$

$$M_d(x_n, \quad y_n,t) = \frac{t}{t + |x_n - y_n|} = \frac{t}{t + \left|\dfrac{x_n - y_n}{2^n}\right|} \text{,then} \quad M_d(x_n, \quad y_n,t) \quad = \quad 1,$$

$|x_n - y_n| \to 0$ as $\quad n \to \quad \infty$.Also $\quad$ in $\quad$ metric $\quad$ space $\quad$ $(\mathbf{R},d)$, $|x_{n+1} - x_n| \leq$

$|x_n - y_n| = \dfrac{|x_0 - y_0|}{2^n}$,if we put

$\dfrac{|x_0 - y_0|}{2^n} = \varepsilon$,then $|x_{n+1} - x_n| \leq \varepsilon$,for every $x_{n+1}, x_n \geq n_0$, $n_0 \in N$ so (x_n) is a Cauchy

sequence in $(\mathbf{R},d)$, then for given $r,t > 0$, $0 < r < 1$,we let $\varepsilon = \dfrac{t}{1-r} - t$,then

$M(x_{n+1}, x_n, t) \geq \dfrac{t}{t + \varepsilon} = 1-r$, for every $x_{n+1}, x_n \geq n_0$ .Thus (x_n) is a Cauchy sequence in

$(\mathbf{R}, M_d, *)$,since $M_d(x_n, y_n, t) = 1$,then $x_n = y_n$ and we let $(x_n) \to x$ in $(\mathbf{R},d)$, then also

$M_d(y_n, x, t) = \dfrac{t}{t + \left|\dfrac{x - y}{2^n}\right|} = 1$,therefore $(y_n) \to x$.

Thus x is boundary point of A .So A is not clopen set. ∎

5-Small inductive dimension(f-ind)

We introduce the following definition in fuzzy metric spaces:

Definition5.1 A fuzzy metric space $X = (X, M, *)$ has small inductive dimension 1 (shortly, F-ind(X) =1) if and only if X is non-zero-dimensional and there is a base $\{B_M(x,r,t) : x \in X; 0 < x < 1, t > 0\}$ for the open sets consisting of set with zero-dimensional boundary.

We prove the following theorem:

Theorem5.2 The fuzzy metric space $R = (R, M, *)$ has a small inductive dimension 1.

Proof. The usual base for the line R consisting of open ball B(x,r,t) which is an open interval or R for the usual topology τ_M. Then the boundary of all such a ball is two – point set, with f-ind dimension is equal to zero. By theorem (4.5) R is not zero-dimensional Thus f-ind(R) = 1. ∎

Example5.3 In the a above theorem if we take M is defined by

$$M(x, y, t) = 1 - \left(\frac{1}{x \wedge y} - \frac{1}{x \vee y}\right)$$ for all x,y $\in R$, t > 0, and $*$ is defined by

$$x*y = \max\{0, x + y - 1\}, \ r < 1/x, \ \text{then} \ B(x,r,t) = \left(\frac{x}{1-rx}, \frac{x}{1+rx}\right),$$ and it is boundary is

two - point set $\left\{\dfrac{x}{1+rx}, \dfrac{x}{1-rx}\right\}$, and f-ind $\left\{\dfrac{x}{1+rx}, \dfrac{x}{1-rx}\right\} = 0$

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On Shrunken Bayesian Estimators for The Mean of Normal Distribution

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Abstract

This paper studies the double-stage shrunken Bayesian estimators (DSSBE) for the mean suggested. In this estimator a shrinkage factor k is taken and the region R was found by minimizing mean squared error. The numerical result shows improvement of the double-stage shrunken Bayesian estimators over the double stage Bayesian estimators in some situations

Introduction

Let x_{ji}, $j = 1,2$, $i = 1,2,...,n_j$, denote two random samples independently normally distributed population with mean θ and variance σ^2 .

Arnold, and Al-Bayyati [4] considered a double stage shrunken estimator of the mean θ when an a prior information about θ is available in the form of an initial estimate θ_0 .

Their estimator is given by:

$$\tilde{\theta} = \begin{cases} k(\hat{\theta}_1 - \theta_0) + \theta_0 & if \ \hat{\theta}_1 \in R \\ \dfrac{n_1\hat{\theta}_1 + n_2\hat{\theta}_2}{n_1 + n_2} & if \ \hat{\theta}_1 \notin R \end{cases} \quad ...(1)$$

Where $0 \le k \le 1$, is the shrinkage factor, R is the parameter, and $\hat{\theta}_1$ is MLE (maximum likelihood estimator) of θ based on n_1 .

Several authors studied the estimator $\tilde{\theta}$ (see, e.g. Whiker, Shurmann, and Raghunath [5], Al-Robassi [1]).

Bayesian method assumes as before that the random sample, $x_1, x_2,...x_n$ can from a population with probability density function $f(x, \theta)$,but further more the unknown parameter θ is a random variable that is there is additional information about θ .

In this paper we study a double stage shrunken Bayesian estimator (DSSBE) of the mean θ of normal distribution when the variance σ^2 is known.

Now suppose that $\hat{\theta}_j = \bar{x}_j$, $j = 1,2$, the sufficient statistic, is the mean of random sample of size n_j; $g(\hat{\theta}_j / \theta)$ is $N(\theta, \dfrac{\sigma^2}{n_j})$. Further more suppose that the prior distribution of θ is defined by:

$$h(\theta) = \frac{1}{\sqrt{2\pi}} \exp(\frac{\theta^2}{2}) \qquad \qquad ...(2)$$

If the square-error loss function $L(\hat{\theta}_j, \theta) = (\hat{\theta}_j - \theta)^2$ is used [3], then the Bayes estimator of θ is given by:

$$\hat{\theta}_{Bj} = \frac{\hat{\theta}_j}{1 + \sigma^2 / n_j} \qquad \qquad ...(3)$$

Therefore a double stage shrunken Bayesian estimator is defined as follows:

$$\tilde{\theta} = \begin{cases} k(\hat{\theta}_{B1} - \theta_0) + \theta_0 & \text{if } \hat{\theta}_{B1} \in R \\[2mm] \dfrac{n_1 \hat{\theta}_{B1} + n_2 \hat{\theta}_{B2}}{n_1 + n_2} & \text{if } \hat{\theta}_{B1} \notin R \end{cases} \qquad ...(4)$$

where, $0 \le k \le 1$, R is the suitable region in the parameter space, and $\hat{\theta}_{B1}$ is the Bayes estimator of θ based on first sample of size n_1.

Mean squared error, Expected sample size, and Efficiency.

In this section the DSSBE of the form (4) is considered, mean squared error, expected sample size and relative efficiency of the estimator $\tilde{\theta}_B$ are derived as follows.

$$MSE(\tilde{\theta}_B / \theta, R) = E(\tilde{\theta}_B - \theta)^2$$
$$= MSE(\hat{\theta}_B) + (k^2 - c_1^2) g_2(\hat{\theta}_{B1} / \theta, R) + [2k(1-k)(\theta_0 - \theta)$$
$$- \frac{2n_1 n_2}{n^2} Bias(\hat{\theta}_{B2})] g_1(\hat{\theta}_{B1} / \theta, R) + [(1-k)^2 (\theta_0 - \theta)^2$$

$$- c_2^2 MSE(\hat{\theta}_{B2})] g_0(\hat{\theta}_{B1} / \theta, R) \qquad \qquad ...(5)$$

where $\hat{\theta}_B = \dfrac{n_1 \hat{\theta}_{B1} + n_2 \hat{\theta}_{B2}}{n_1 + n_2}$, $\quad c_1^2 = \dfrac{n_1^2}{n^2}$, $\quad c_2^2 = \dfrac{n_2^2}{n^2}$, $\quad n = n_1 + n_2$,

and $g_i(\hat{\theta}_{B1} / \theta, R) = \int_R (\hat{\theta}_{B1} - \theta)^i f(\hat{\theta}_{B1}) d(\hat{\theta}_{B1})$, $i = 1,2$

If θ_0 is the true value of θ then equation (5) becomes:

$$MSE(\tilde{\theta}_B / \theta_0, R) = E(\tilde{\theta}_B - \theta_0)^2$$
$$= MSE(\hat{\theta}_B) + (k^2 - c_1^2) g_2(\hat{\theta}_{B1} / \theta_0, R)$$
$$- \frac{2n_1 n_2}{n^2} Bias(\hat{\theta}_{B2}) g_1(\hat{\theta}_{B1} / \theta_0, R)$$

$$- c_2^2 MSE(\hat{\theta}_{B2}) g_0(\hat{\theta}_{B1} / \theta_0, R) \qquad \qquad \ldots(6)$$

Let us choose the region R so that $MSE(\theta_B / \theta_0, R)$ is minimum (see [4]) we get :

$$R = \theta_0 + \frac{n_1 n_2 Bias(\hat{\theta}_{B1}) - a}{n^2 (k^2 - c_1^2)}, \theta_0 + \frac{n_1 n_2 Bias(\hat{\theta}_{B1}) + a}{n^2 (k^2 - c_1^2)} \qquad (7)$$

where $a = \sqrt{c_1^2 c_2^2 [Bias(\hat{\theta}_{B2})]^2 + (k^2 - c_1^2) MSE(\hat{\theta}_{B2})}$.

The expected sample size is given by:

$$E(n / \theta, R) = n - (n - n_1) P_r (\hat{\theta}_{B1} \in R)$$

$$= n - (n - n_1) g_0 (\hat{\theta}_{B1} / \theta, R) \qquad \qquad \ldots(8)$$

Therefore, we define the efficiency of the estimator $\tilde{\theta}_B$ with respect to the Bayesian estimator $\hat{\theta}_B$ by:

$$Eff(\tilde{\theta}_B / \theta, R) = \frac{MSE(\hat{\theta}_B, based\ on\ a\ sample\ of\ equivalent\ size)}{MSE(\tilde{\theta}_B / \theta, R)} \qquad \ldots(9)$$

where $\hat{\theta}_B = \frac{n_1 \hat{\theta}_{B1} + n_2 \hat{\theta}_{B2}}{n_1 + n_2}$.

Numerical results

The computation of mean squared error and efficiency of the estimator $\tilde{\theta}_B$ with respect to the Bayes estimator $\hat{\theta}_B$ considered in this section by taking

$$n = 50, \ n_1 = 5, 10, 15, 20, \ k = 0.25, \ 0.5, \ 0.75 \ and \ t = \sqrt{n_1} \frac{|\theta - \theta_0|}{\sigma} .$$

The following conclusions are based on these computations :

(1) The probability of avoiding the second sample , is decreasing function of the shrinkage factor, whereas it is increasing function of the first sample n_1 (see table (1)).

(2) Mean squared error of $\widetilde{\theta}_B$ is increasing function of the shrinkage factor, and decreasing function with the first sample n_1 (see table (2)).

(3) As expected, the efficiency of the double stage shrunken Bayesian estimator $\widetilde{\theta}_B$, is better than the efficiency of the Bayesian estimator $\hat{\theta}_B$,when θ is close to θ_0 .

(4) Table (3) indicates that the efficiency of the suggested estimator is decreasing function of the shrinkage factor k , and increasing function of n_1.

(5) All value of k and n_1 give highest efficiency only in neighborhood t=0 .

Table(1)

Probability Of avoiding the second sample for, $n = 50, \ \theta = 10, \ \sigma = 5, \ t = \dfrac{|\theta - \theta_0|}{\sigma} \sqrt{n_1}$

	$n_1 = 5$			$n_1 = 10$			$n_1 = 15$			$n_1 = 20$		
k t	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0.0	0.938	0.823	0.761	0.959	0.873	0.837	0.962	0.890	0.871	0.969	0.913	0.892
0.2	0.917	0.786	0.710	0.936	0.859	0.791	0.943	0.875	0.830	0.947	0.891	0.864
0.4	0.895	0.737	0.663	0.918	0.839	0.739	0.927	0.858	0.811	0.929	0.875	0.830
0.6	0.872	0.694	0.624	0.889	0.778	0.701	0.903	0.826	0.775	0.912	0.850	0.801
0.8	0859	0.657	0.562	0.867	0.733	0.671	0.889	0.783	0.729	0.896	0.821	0.771
1.0	0.847	0.615	0.519	0.849	0.692	0.604	0.868	0.749	0.683	0.871	0.785	0.729
1.2	0819	0.572	0.468	0.826	0.657	0.569	0.841	0.709	0.623	0.853	0.736	0.670
1.4	0.789	0.529	0.419	0.810	0.617	0.520	0.829	0.673	0.560	0.837	0.706	0.612
1.6	0.764	0.481	0.361	0.795	0.561	0.482	0.815	0.660	0.516	0.820	0.673	0.564
1.8	0.735	0.428	0.321	0.774	0.510	0.441	0.790	0.569	0.469	0.805	0.626	0.521
2.0	0.697	0.357	0.211	0.730	0.483	0.387	0.779	0.517	0.413	0.789	0.580	0.460

Table(2)

$$\text{MSE}(\widetilde{\theta}_B \,/\, \theta, R) \ \text{for,} \ \theta = 10, \ \sigma = 5, \ and \ t = \dfrac{|\theta - \theta_0|}{\sigma} \sqrt{n_1}$$

$n_1 = 5$ $\qquad\qquad\qquad$ $n_1 = 10$ $\qquad\qquad\qquad$ $n_1 = 15$ $\qquad\qquad\qquad$ $n_1 = 20$

k	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0.0	0.099	0.158	0.171	0.061	0.097	0.117	0.046	0.89	0.108	0,043	0.087	0.103
0.2	0.118	0.169	0.186	0.073	0.118	0.128	0.057	0.115	0.126	0.051	0.113	0.123
0.4	0.141	0.182	0.203	0.097	0.133	0.143	0.071	0.128	0.139	0.078	0.126	0.137
0.6	0.185	0.196	0.221	0.113	0.155	0.164	0.103	0.147	0.157	0.095	0.146	0.155
0.8	0.230	0.208	0.237	0.129	0.178	0.181	0.121	0.173	0.176	0.117	0.170	0.175
1.0	0.279	0.225	0.251	0.141	0.195	0.204	0.149	0.192	0.198	0.141	0.189	0.187
1.2	0.331	0.291	0.304	0.169	0.237	0.233	0.169	0.219	0.223	0.163	0.214	0.221
1.4	0.398	0.312	0.327	0.182	0.261	0.274	0.187	0.247	0.257	0.184	0.239	0.243
1.6	0.461	0.326	0.339	0.227	0.315	0.301	0.219	0.293	0.294	0.215	0.270	0.280
1.8	0.365	0.355	0.348	0.251	0.341	0.356	0.249	0.330	0.330	0.247	0.312	0.327
2.0	0.571	0.458	0.427	0.289	0.407	0.419	0.287	0.387	0.379	0.278	0.358	0.368

Table(3)

$$Eff\,(\widetilde{\theta}_B\,/\,\theta, R) \text{ for, } \theta = 10, \ \sigma = 5, \text{ and } t = \frac{|\theta - \theta_0|}{\sigma}\sqrt{n_1}$$

$n_1 = 5$ $n_1 = 10$ $n_1 = 15$ $n_1 = 20$

k t	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0.0	6.526	4.011	2.953	7.821	4.189	3.413	8.201	5.203	3.932	8.900	5.813	4.310
0.2	5.703	3.315	2.341	6.395	3.507	2.971	7.053	4.047	3.045	8.015	4.965	3.591
0.4	4.250	2.712	1.932	4.395	2.915	2.576	5.917	3.341	2.610	7.217	4.072	2.843
0.6	3.801	1.903	1.057	3.317	2.014	1.789	4.731	2.865	1.969	6.044	3.001	2.169
0.8	2.753	1.142	0.980	2.650	1.581	1.316	3.021	2.054	1.497	4.351	2.379	1.629
1.0	1.830	0.873	0.789	1.813	0.899	0.983	2.450	1.631	1.000	3.190	1.832	1.110
1.2	1.012	0.705	0.703	1.259	0.758	0.837	1.534	0.971	0.872	2.418	1.006	0.901
1.4	0.831	0.571	0.639	0.985	0.696	0.721	1.069	0.823	0.763	1.505	0.879	0.829
1.6	0.679	0.485	0.561	0.739	0.587	0.649	0.869	0.765	0.703	0.892	0.781	0.740
1.8	0.485	0.445	0.483	0.590	0.569	0.573	0.753	0.682	0.649	.782	0.718	0.683
2.0	0.401	0.406	0.415	0.543	0.550	0.525	0.614	0.603	0.583	0.631	0.662	0.616

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مقدرات التقلص البيزية
لتقدير الوسط الحسابي للتوزيع الطبيعي

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الملخص

بفـرض أن x متغيراً عشـوائياً يتبـع التوزيـع الطبيعـي بوسـط حسـابي مجهـول وتبـاين معلـوم في حالـة تـوافر معلومـات مسـبقة حـول المعلمـة المجهولـة علـى شـكل قيمـة ابتدائيـة يكـون من المفيد استخدام مقدرات التقلص ذات المرحلتين لتقدير المعلمة المجهولة إنظر (2) .

تناولـت هـذه الدراسـة مقـدرات التقلـص البيزيـة ذات المـرحلتين لتقـدير الوسـط الحسـابي للتوزيـع الطبيعـي لقـيم مختلفـة لعامـل التقلص K ، كمـا تم اختيـار المجـال R بواسـطة تصغير متوسط مربعات الخطأ ، وقـد بينـت النتـائج العدديـة أن المقـدر المقتـرح يمتلـك متوسـط مربعات خطأ أقل من مقدرات البيزية وبالتالي كفاءة نسبية أعلى .