

THE RIEMANN-HILBERT PROBLEM AND THE GENERALIZED NEUMANN KERNEL ON UNBOUNDED MULTIPLY CONNECTED REGIONS

Mohamed M.S. Nasser

*Department of Mathematics, Faculty of Science, Ibb University,
P. O. Box 70270, Ibb, Yemen, E-mail: mms_nasser@hotmail.com*

Abstracts. A Fredholm integral equation of the second kind with the generalized Neumann kernel associated with the Riemann-Hilbert problem on unbounded multiply connected regions will be derived and studied in this paper. The derived integral equation yields a uniquely solvable boundary integral equations for the modified Dirichlet problem on unbounded multiply connected regions.

Keywords. Riemann-Hilbert problem; generalized Neumann kernel.

Mathematical Subject Classification. 30E25; 45B05.

1) Introduction

The interplay of Riemann-Hilbert problems (RH problems, for short) and integral equations with the generalized Neumann kernel has been investigated in [12] for simply connected regions, in [13] for bounded multiply connected regions, and in [9] for simply connected regions with piecewise smooth boundaries. Integral equations with the generalized Neumann kernel on bounded multiply connected regions have been used in [7] to develop a unified method for computing conformal mapping onto the classical slit domains. Based on the results of this paper, the method presented in [7] can be extended to unbounded multiply connected regions (see [8]).

In this paper, we shall extend the results of [12, 13] to unbounded multiply connected regions. We derive a second kind Fredholm integral equation with the generalized Neumann kernel for the RH problems on unbounded multiply connected regions. Then, based on a Moebius transform, the properties of the derived integral equation will be deduced from the properties of integral equations with the generalized Neumann kernel on bounded multiply connected regions. The derived integral equation yields a uniquely solvable boundary integral equation for the modified Dirichlet problem, which is a special case of the RH problem.

The remainder of this paper is organized as follows: we present some auxiliary material in Section 2. Section 3 presents an integral equation with the generalized

Neumann kernel for the RH problem on unbounded multiply connected regions. The solvability of the derived integral equation will be studied in Section 4. Section 5 presents a uniquely solvable integral equation for the modified Dirichlet problem. Finally, short conclusions will be given in Section 6.

2 Auxiliary material

Suppose that G is an unbounded multiply connected region of connectivity m bounded by $\Gamma := \partial G = \Gamma_1 \cup \Gamma_2 \cup \dots \cup \Gamma_m$ where the curves Γ_j , $j = 1, 2, \dots, m$, are simple non-intersecting smooth clockwise oriented closed curves (see Figure 1). We assume that $\infty \in G$ and $0 \in G$. The complement $G^- := (\mathbb{C} \cup \{\infty\}) \setminus (G \cup \Gamma)$ of G consists of m bounded simply connected regions G_j , $j = 1, 2, \dots, m$. The curve Γ_j is parametrized by a 2π -periodic twice continuously differentiable complex function $\eta_j(s)$ which transverses Γ_j in the clockwise orientation with $\dot{\eta}_j(s) = d\eta_j(s)/ds \neq 0$. The total parameter domain J is the disjoint union of m intervals $J_j := [0, 2\pi]$, $j = 1, 2, \dots, m$. We define a parametrization of the whole boundary as the complex function η defined on J by

$$\eta(s) := \begin{cases} \eta_1(s), & s \in J_1, \\ \vdots \\ \eta_m(s), & s \in J_m. \end{cases} \quad (1)$$



Figure 1: An unbounded multiply connected region G of connectivity m .

Let H be the space of all real Holder continuous 2π -periodic functions $\phi(s)$ of the parameter s on J_j for $j = 1, 2, \dots, m$, i.e.,

$$\phi(s) := \begin{cases} \phi_1(s), & s \in J_1, \\ \vdots \\ \phi_m(s), & s \in J_m, \end{cases}$$

where ϕ_1, \dots, ϕ_m are real Holder continuous 2π -periodic functions. In view of the smoothness of η , a real Holder continuous function $\hat{\phi}$ on Γ can be interpreted via $\phi(s) := \hat{\phi}(\eta(s))$ as a function $\phi \in H$; and vice versa. If $\phi \in H$ is a piecewise constant real-valued function, i.e.,

$$\phi(s) = \begin{cases} \alpha_1, & s \in J_1, \\ \vdots \\ \alpha_m, & s \in J_m, \end{cases}$$

with real constants $\alpha_1, \dots, \alpha_m$, then ϕ will be denoted by

$$\phi(s) = (\alpha_1, \dots, \alpha_m).$$

Let A be a continuously differentiable complex function on Γ with $A \neq 0$. We assume that A is given in the parametric form $A(s)$ such that $A(s)$ is continuously differentiable for all $s \in J$. With $\gamma, \mu \in H$, we define the function

$$\Phi(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\gamma + i\mu}{A} \frac{d\eta}{\eta - z}, \quad z \notin \Gamma. \tag{2}$$

Then Φ is an analytic function in G as well as in G^- with $\Phi(\infty) = 0$. The boundary values Φ^+ from inside G (the left of Γ) and Φ^- from outside G (the right of Γ) are Holder continuous on Γ and can be calculated by Plemelj's formulas

$$\Phi^{\pm}(\eta(s)) = \pm \frac{1}{2} \frac{\gamma(s) + i\mu(s)}{A(s)} + \frac{1}{2\pi i} \int_J \frac{\gamma(t) + i\mu(t)}{A(t)} \frac{\eta(t) dt}{\eta(t) - \eta(s)}. \tag{3}$$

The integral in (3) is a Cauchy principal value integral. It follows from (3) that the boundary values satisfy the jump relation

$$A(s)\Phi^+(\eta(s)) - A(s)\Phi^-(\eta(s)) = \gamma(s) + i\mu(s). \tag{4}$$

Interior RH problem: Search a function f analytic in G with $f(\infty) = 0$, continuous on $G \cup \Gamma$, such that the boundary values f^+ satisfy on Γ

$$\operatorname{Re}[A(s)f^+(\eta(s))] = \gamma(s). \tag{5}$$

To extend the results of [12, 13] to unbounded multiply connected regions, we shall borrow some definitions and notations from [12, 13]. We define the following boundary value problem in the exterior region G^- as the exterior RH problem.

Exterior RH problem: Search a function g analytic in G^- , continuous on $G^- \cup \Gamma$, such that the boundary values g^- satisfy on Γ

$$\operatorname{Re}[A(s)g^-(\eta(s))] = \gamma(s). \quad (6)$$

We define also the range spaces R^\pm as the spaces of all real functions $\gamma \in H$ for which the RH problems are solvable and the spaces S^\pm as the spaces of the boundary values of solutions of the homogeneous RH problems, i.e.,

$$R^+ := \{\gamma \in H: \gamma = \operatorname{Re}[Af^+], f \text{ analytic in } G, f(\infty) = 0\}, \quad (7)$$

$$S^+ := \{\gamma \in H: \gamma = Af^+, f \text{ analytic in } G, f(\infty) = 0\}, \quad (8)$$

$$R^- := \{\gamma \in H: \gamma = \operatorname{Re}[Ag^-], g \text{ analytic in } G^-\}, \quad (9)$$

$$S^- := \{\gamma \in H: \gamma = Ag^-, g \text{ analytic in } G^-\}. \quad (10)$$

3 The generalized Neumann kernel

We define the real kernels M and N as real and imaginary parts (see [12, 13] for details)

$$M(s, t) + iN(s, t) := \frac{1}{\pi} \frac{A(s)}{A(t)} \frac{\dot{\eta}(t)}{\eta(t) - \eta(s)}. \quad (11)$$

The kernel $N(s, t)$ is called the *generalized Neumann kernel* formed with A and η . It is continuous with

$$N(t, t) = \frac{1}{\pi} \operatorname{Im} \left(\frac{1}{2} \frac{\dot{\eta}(t)}{\dot{\eta}(t)} - \frac{\dot{A}(t)}{A(t)} \right). \quad (12)$$

When $s, t \in J_j$ are in the same parameter interval J_j , then

$$M(s, t) = -\frac{1}{2\pi} \cot \frac{s-t}{2} + M_1(s, t) \quad (13)$$

with a continuous kernel M_1 which takes on the diagonal the values

$$M_1(s, t) = \frac{1}{\pi} \operatorname{Re} \left(\frac{1}{2} \frac{\dot{\eta}(t)}{\dot{\eta}(t)} - \frac{\dot{A}(t)}{A(t)} \right). \quad (14)$$

We define the integral operators with the kernels \mathbf{N} and \mathbf{M} by

$$\mathbf{N}\mu(s) := \int_J N(s, t)\mu(t)dt, \quad s \in J, \quad (15)$$

$$\mathbf{M}\mu(s) := \int_J M(s, t)\mu(t)dt, \quad s \in J. \quad (16)$$

The operator \mathbf{N} is a Fredholm integral operator and \mathbf{M} is a singular operator where the integral in (16) is a principal value integral.

Lemma 1. (a) The boundary values of the function Φ defined in (2) can be represented in terms of the operators \mathbf{N} and \mathbf{M} by

$$2A\Phi^\pm = (\pm\mathbf{I} + \mathbf{N} - i\mathbf{M})(\gamma + i\mu). \quad (17)$$

Proof: By multiplying both sides of (3) by $2A(\zeta)$ and using the definitions of the operators \mathbf{N} and \mathbf{M} , we obtain (17). ■

Lemma 2. (a) If f is analytic function in the unbounded region G with $f(\infty) = 0$, then

$$(\mathbf{I} - \mathbf{N} + i\mathbf{M})(Af^+) = 0. \quad (18)$$

(b) If g is analytic function in the bounded region G^- , then

$$(\mathbf{I} + \mathbf{N} - i\mathbf{M})(Ag^-) = 0. \quad (19)$$

Proof: (a) Let $\gamma := \text{Re}[Af^+]$, $\mu := \text{Im}[Af^+]$ and Φ be formed with γ, μ according to (2). By the Cauchy integral formula, we have $\Phi = f$ in G , which implies that, $A\Phi^+ = Af^+ = \gamma + i\mu$. Hence, (18) follows from (17).

(b) Let $\gamma := \text{Re}[Ag^-]$, $\mu := \text{Im}[Ag^-]$ and Φ be formed with γ, μ according to (2). Since G^- is on the right of Γ , the Cauchy integral formula implies that $\Phi = -g$ in G^- . Hence, $A\Phi^- = -Ag^- = -(\gamma + i\mu)$ and (19) follows from (17). ■

There is a close connection between RH problems and integral equations with the generalized Neumann kernel (see also [9, 12, 13]).

Theorem 1. If f is a solution of the RH problem (5) with boundary values

$$Af^+ = \gamma + i\mu, \quad (20)$$

then the imaginary part μ in (20) satisfies the integral equation

$$\mu - \mathbf{N}\mu = -\mathbf{M}\gamma. \quad (21)$$

Proof: Substituting (20) into (18) then taking the imaginary part, we obtain (21). ■

It follows from the previous theorem that a solution of the RH problem (5) yields a solution of the integral equation (21). To use the integral equation (21) to solve the RH problem (5), we have the following theorem.

Theorem 2. Let the real function $\gamma \in H$ be given, μ be a solution of (21) and Φ be formed with γ, μ by (2). Then $f := \Phi$ in G satisfies

$$Af^+ = \gamma + h + i\mu \quad (22)$$

With

$$h = [\mathbf{M}\mu - (I - \mathbf{N})\gamma]/2.$$

Proof: The jump relation (4) implies that the function $f := \Phi$ in G has the boundary values

$$Af^+ = A\Phi^+ = A\Phi^- + \gamma + i\mu.$$

Since μ satisfies the integral equation (21), it follows from (17) that

$$2A\Phi^- = -\gamma - \mathbf{N}\gamma + \mathbf{M}\mu$$

which implies that $h := A\Phi^-$ is real and given by (23). Hence, $h \in S^-$. ■

4 The solvability

The solvability of RH problems on bounded multiply connected regions was discussed by Vekua [10] (see also [2, 14]). The solvability on unbounded multiply connected regions can be deduced from the solvability on bounded regions by means of Moebius transform (see e.g., [11, p. 141]). Let z_0 be a fixed point in G^- , say $z_0 \in G_m$. The unbounded region G is transformed by means of the Moebius transform

$$\Psi(z) := \frac{1}{z - z_0} \quad (24)$$

onto a bounded multiply connected region $\hat{G} := \Psi(G)$ of connectivity m . The Moebius transform Ψ transforms also the bounded exterior region G^- onto an unbounded region \hat{G}^- exterior to the boundary $\hat{\Gamma} := \partial\hat{G} = \Psi(\Gamma)$. The boundary $\hat{\Gamma}$ is given by $\hat{\Gamma} := \hat{\Gamma}_0 \cup \hat{\Gamma}_1 \cup \hat{\Gamma}_{m-1}$ where $\hat{\Gamma}_0 := \Psi(\Gamma_m)$ is the outer curve and is counterclockwise oriented; and the other curves $\hat{\Gamma}_j := \Psi(\Gamma_j)$, $j = 1, 2, \dots, m-1$, are clockwise oriented and are inside $\hat{\Gamma}_0$. The curve $\hat{\Gamma}$ is parametrized by

$$\zeta(s) := \frac{1}{\eta(s) - z_0}, \quad s \in J. \quad (25)$$

Let the function \hat{A} be defined by

$$\hat{A}(s) := \zeta(s)A(s), \quad s \in J. \quad (26)$$

Then, for a given function $\gamma \in H$, we define the RH problems in the new regions \hat{G} and \hat{G}^- as follows:

Interior RH problem: Search a function \hat{f} analytic in \hat{G} , continuous on $\hat{G} \cup \hat{\Gamma}$, such that the boundary values \hat{f}^+ satisfy on $\hat{\Gamma}$

$$\text{Re}[\hat{A}(s) \hat{f}^+(\zeta(s))] = \gamma(s). \quad (27)$$

Exterior RH problem: Search a function \hat{g} analytic in \hat{G}^- with $\hat{g}(\infty) = 0$, continuous on $\hat{G} \cup \hat{\Gamma}$, such that the boundary values \hat{g}^- satisfy on $\hat{\Gamma}$

$$\text{Re}[\hat{A}(s) \hat{g}^-(\zeta(s))] = \gamma(s). \quad (28)$$

We define also the spaces

$$\hat{R}^+ := \{\gamma \in H: \gamma = \text{Re}[\hat{A}\hat{f}^+], \hat{f} \text{ analytic in } \hat{G}\}, \quad (29)$$

$$\hat{S}^+ := \{\gamma \in H: \gamma = \hat{A}\hat{f}^+, \hat{f} \text{ analytic in } \hat{G}\}, \quad (30)$$

$$\hat{R}^- := \{\gamma \in H: \gamma = \text{Re}[\hat{A}\hat{g}^-], \hat{g} \text{ analytic in } \hat{G}^-, \hat{g}(\infty) = 0\}, \quad (31)$$

$$\hat{S}^- := \{\gamma \in H: \gamma = \hat{A}\hat{g}^-, \hat{g} \text{ analytic in } \hat{G}^-, \hat{g}(\infty) = 0\}. \quad (32)$$

Lemma 3. (a) A function f is a solution of the RH problem(5) in the unbounded region G if and only if the function

$$\hat{f}(w) := \frac{(f \circ \Psi^{-1})(w)}{w} \quad (33)$$

is a solution of the RH problem (27) in the bounded region \widehat{G} .

(b) A function g is a solution of the exterior RH problem (6) in the bounded region G^- if and only if the function

$$\widehat{g}(w) := \frac{(g \circ \Psi^{-1})(w)}{w} \quad (34)$$

is a solution of the RH problem (28) in the unbounded region \widehat{G}^- .

Proof: The proof follows from the definition of the Moebius transform Ψ and the definitions of the RH problems (5), (6), (27), (28). ■

Lemma 4. The spaces $R^\pm, S^\pm, \widehat{R}^\pm$ and \widehat{S}^\pm satisfy

$$\widehat{R}^+ = R^+, \quad \widehat{S}^+ = S^+, \quad \widehat{R}^- = R^-, \quad \widehat{S}^- = S^-. \quad (35)$$

Proof: In view of Lemma 3, the proof follows from the definitions of the spaces $R^\pm, S^\pm, \widehat{R}^\pm$ and \widehat{S}^\pm . ■

The index κ_j of the function A on the curve Γ_j is defined as the winding number of A with respect to 0,

$$\kappa_j := \frac{1}{2\pi} \Delta \arg(A)|_{\Gamma_j}, \quad j = 1, 2, \dots, m, \quad (36)$$

i.e., the change of the argument of A along the curve Γ_j divided by 2π . The index κ of the function A on the whole boundary curve Γ is the sum

$$\kappa := \sum_{j=1}^m \kappa_j. \quad (37)$$

The index $\widehat{\kappa}_j$ of the function \widehat{A} on the curve $\widehat{\Gamma}_j$ and the index $\widehat{\kappa}$ of the function \widehat{A} on the whole boundary $\widehat{\Gamma}$ are easily calculated from the index of the function A by

$$\widehat{\kappa}_0 = \kappa_m + 1, \quad \widehat{\kappa}_j = \kappa_j, \quad j = 1, 2, \dots, m-1, \quad \widehat{\kappa} = \kappa + 1. \quad (38)$$

The RH problems (27) and (28) are of the types studied in [13]. Thus, in view of Lemma 4 and Eq. (38), we have the following results from [13] for the solvability of the RH problems (5) and (6).

Lemma 5 [13]. The spaces R^- and S^\pm satisfy

$$S^- \cap S^+ = \{0\}, \tag{39}$$

$$R^- \cap S^+ = \{0\}. \tag{40}$$

Theorem 3 ([13]). The dimension of the space S^- and the codimension of the space R^- are determined by the index of A as follows:

$$\dim(S^-) = \sum_{j=1}^m \max(0, 2\kappa_j + 1), \tag{41}$$

$$\text{codim}(R^-) = \sum_{j=1}^m \max(0, -2\kappa_j - 1). \tag{42}$$

Theorem 4 ([13]). The dimension of the space S^+ and the codimension of the space R^+ are determined by the index of A as follows:

(a) If $\kappa \geq 0$, then

$$\dim(S^+) = 0, \quad \text{codim}(R^+) = 2\kappa + m. \tag{43}$$

(b) If $\kappa \leq -m$, then

$$\dim(S^+) = -2\kappa - m, \quad \text{codim}(R^+) = 0. \tag{44}$$

(c) If $-m + 1 \leq \kappa \leq -1$, then

$$-2\kappa - m \leq \dim(S^+) \leq -\kappa, \quad 2\kappa + m \leq \text{codim}(R^+) \leq m + \kappa. \tag{45}$$

For studying the solvability of the integral equation (21), it follows from (25) that

$$\eta(s) = \frac{1}{\zeta(s)} + z_0, \quad \dot{\eta}(s) = -\frac{\dot{\zeta}(s)}{\zeta(s)^2}, \quad s \in J.$$

Hence

$$\frac{A(s)}{A(t)} \frac{\dot{\eta}(t)}{\eta(t) - \eta(s)} = \frac{\hat{A}(s)}{\hat{A}(t)} \frac{\dot{\zeta}(t)}{\zeta(t) - \zeta(s)} \tag{46}$$

where \hat{A} is defined by (26). Let the real kernels \hat{M} and \hat{N} be defined by

$$\widehat{M}(s, t) + i\widehat{N}(s, t) := \frac{\widehat{A}(s)}{\widehat{A}(t)} \frac{\zeta(t)}{\zeta(s)}. \quad (47)$$

i.e., the kernel \widehat{N} is the generalized Neumann kernel formed with \widehat{A} and ζ . Let $\widehat{\mathbf{M}}$ and $\widehat{\mathbf{N}}$ be the integral operators defined on H with the kernels \widehat{M} and \widehat{N} . In view of (46), we have $\widehat{N}(s, t) = N(s, t)$ and $\widehat{M}(s, t) = M(s, t)$ for all $(s, t) \in J \times J$. Hence

$$\widehat{\mathbf{N}} = \mathbf{N} \quad \text{and} \quad \widehat{\mathbf{M}} = \mathbf{M}. \quad (48)$$

The operators $\widehat{\mathbf{M}}$ and $\widehat{\mathbf{N}}$ are of the types studied in [13]. Thus, in view of Lemma 4 and Eq. (38), we have the following results from [13] for the properties of the integral operators \mathbf{N} and \mathbf{M} .

Lemma 6 ([13]). The operators \mathbf{N}, \mathbf{M} and the identity operator \mathbf{I} are connected by the following relations:

$$\mathbf{N}^2 - \mathbf{M}^2 = \mathbf{I}, \quad (49)$$

$$\mathbf{NM} + \mathbf{MN} = 0. \quad (50)$$

Theorem 5 ([13]). The range-spaces and the null-spaces of the operators \mathbf{M} and $\mathbf{I} \pm \mathbf{N}$ are related to the spaces S^\pm and R^\pm by

$$\text{Range}(\mathbf{M}) = R^+ \cap R^-, \quad (51)$$

$$\text{Null}(\mathbf{M}) = S^+ \oplus S^-, \quad (52)$$

$$\text{Range}(\mathbf{I} + \mathbf{M}) \subset R^+, \quad (53)$$

$$\text{Null}(\mathbf{I} + \mathbf{M}) = S^-, \quad (54)$$

$$\text{Range}(\mathbf{I} - \mathbf{M}) = R^-, \quad (55)$$

$$\text{Null}(\mathbf{I} - \mathbf{M}) = S^+ \oplus W, \quad (56)$$

where W is isomorphic (via \mathbf{M}) to $R^+ \cap S^-$.

Corollary 1. The integral equation (21) is solvable for all $\gamma \in H$.

Proof: The Formulas (51) and (55) imply that

$$\text{Range}(\mathbf{M}) \subset \text{Range}(\mathbf{I} - \mathbf{N}),$$

which implies that the integral equation (21) is always solvable. ■

Theorem 6 ([13]). The dimensions of the null-spaces of the operators $\mathbf{I} \pm \mathbf{N}$ are given by

$$\dim(\text{Null}(\mathbf{I} + \mathbf{N})) = \sum_{j=1}^m \max(0, 2\kappa_j + 1), \tag{57}$$

$$\dim(\text{Null}(\mathbf{I} - \mathbf{N})) = \sum_{j=1}^m \max(0, -2\kappa_j - 1). \tag{58}$$

Lemma 7 ([13]). If λ is an eigenvalue of \mathbf{N} with eigenfunction $v \notin S^+ \cup S^-$, then $-\lambda$ is also an eigenvalue of \mathbf{N} and $\mathbf{M}v$ is a corresponding eigenfunction.

It follows from Theorem 4 that the RH problem (5) is not necessary solvable for general A and γ . In the following theorem, we shall show that the right-hand side of (5) can be modified such that the new problem is solvable.

Theorem 7. For any $\gamma \in H$, there exists a real function $h \in S^-$ such that the following RH problem

$$\text{Re}[Af^+] = \gamma + h \tag{59}$$

is solvable.

Proof: For any $\gamma \in H$, it follows from Corollary 1 that the integral equation (21) has a solution μ . Then, Theorem 2 implies that a real function $h := [\mathbf{M}\mu - (\mathbf{I} - \mathbf{N})\gamma]/2 \in S^-$ exists such that $Af^+ = \gamma + h + i\mu$ are boundary values of an analytic solution f in G . Hence, f is a solution of the RH problem (59). ■

Theorem 8. If $\dim(S^+) = \dim(\mathbf{I} - \mathbf{N})$, then the space H has the decomposition

$$H = R^+ \oplus S^-. \tag{60}$$

Proof: Let $\gamma \in H$ be a given function. It follows from Theorem 7 that a function $h \in S^-$ exists such that $\gamma + h \in R^+$. Hence

$$H = R^+ + S^-. \tag{61}$$

Since $\dim(S^+) = \dim(\mathbf{I} - \mathbf{N})$, it follows from (56) that

$$\dim(R^+ \cap S^-) = \dim(\text{Null}(\mathbf{I} - \mathbf{N})) - \dim(S^+) = 0,$$

which implies that $R^+ \cap S^- = \{0\}$. Hence, the sum in (61) is direct. ■

Corollary 2. If $\dim(S^+) = \dim(\mathbf{I} - \mathbf{N})$, then for any $\gamma \in H$, there exists a unique function $h \in S^-$ such that the RH problem (59) is solvable.

5 The Modified Dirichlet problem

In this section, we shall study the solvability the RH problem (59) and the integral equation (21) for the special case

$$A = 1. \quad (62)$$

In this case, the kernel \mathbf{N} is known as the Neumann kernel [3, p.286] and the RH problem (5) is known as the modified Dirichlet problem [2, 5, 6, 14] or the Schwartz problem [2]. This special case is of practical use in conformal mapping of unbounded multiply connected regions (see [8]).

The function $A = 1$ has the index

$$\kappa_j = 0, \quad j = 1, 2, \dots, m, \quad \kappa = 0. \quad (63)$$

Hence, Theorems 3, 4 and 6 imply that

$$\dim(S^-) = m, \quad \dim(S^+) = 0, \quad \dim(\text{Null}(\mathbf{I} - \mathbf{N})) = 0. \quad (64)$$

Let the real function $\chi^{[j]}(s)$, $j = 1, 2, \dots, m$, be defined for $s \in J$ by

$$\chi^{[j]}(s) := \begin{cases} 1, & s \in J_j, \\ 0, & s \notin J_j, \end{cases} \quad (65)$$

and the sectionally analytic function $g^{[j]}(z)$ be defined by

$$g^{[j]}(z) := \begin{cases} 1, & z \in G_j, \\ 0, & z \text{ in the exterior domain of the curve } \Gamma_j. \end{cases} \quad (66)$$

Hence $\chi^{[j]} = Ag^{[j]} \in S^-$ for all $j = 1, 2, \dots, m$. From (64), we have $\dim(S^-) = m$. Since $\chi^{[1]}, \dots, \chi^{[m]}$ are linearly independent, we obtain

$$S^- = \text{span}\{\chi^{[1]}, \dots, \chi^{[m]}\}. \quad (67)$$

Theorem 9. For any $\gamma \in H$, there exists a unique solution μ of the integral equation (21) and a unique function $h = (h_1, \dots, h_m)$ given by (23) such that $f^+ = \gamma + h + i\mu$ are boundary values of the unique solution of the RH problem (59).

Proof. By (64), $\text{Null}(\mathbf{I} - \mathbf{N}) = \{0\}$ which implies, in view of Fredholm alternative theorem, that the integral equation (21) has a unique solution μ . Then, Theorem 2 implies that $f^+ = \gamma + h + i\mu$ are boundary values of an analytic function f in G where $h \in S^-$ is given by (23). Since $\dim(S^+) = 0$, f is the unique solution of the RH problem (59). In view of Eq. (64), Corollary 2 and Eq. (67) imply that the function h is unique and $h = (h_1, \dots, h_m)$ with real constants h_1, \dots, h_m . ■

7) Conclusions

We have derived and studied a boundary integral equation with the generalized Neumann kernel for the RH problem on unbounded multiply connected regions. By means of a Moebius transform, we obtained the solvability of the derived integral equation from the related known results for bounded regions. Then, the derived integral equation was used to obtain boundary integral equations for the modified Dirichlet problem on unbounded multiply connected regions.

The boundary integral equations were derived in this paper for regions with smooth boundaries. Nevertheless, these equations, with slight modifications, can be applied to regions with corners (see [9] for simply connected regions case).

Several accurate numerical methods are available for solving boundary integral equations (see e.g. [1]). For regions with smooth boundary, one can use the Nystrom method with the trapezoidal rule [1]. For regions with corners, we can use a Nystrom method based on the trapezoidal rule with a graded mesh [1,4].

References

- [1] K.E. Atkinson, *The Numerical Solution of Integral Equations of the Second Kind*, Cambridge University Press, Cambridge, 1997.
- [2] F.D. Gakhov, *Boundary Value Problem*, English translation of Russian edition 1963, Pergamon Press, Oxford, 1966.
- [3] P. Henrici, *Applied and Computational Complex Analysis*, Vol. 3, John Wiley, New York, 1986.
- [4] R. Kress, A Nystrom method for boundary integral equations in domains with corners, *Numer. Math.* 58 (1990) 145-161.
- [5] S.G. Mikhlin, *Integral Equations*, English translation of Russian edition 1948, Pergamon Press, Armstrong, 1957.
- [6] N.I. Muskhelishvili, *Singular Integral Equations*, English translation of Russian edition 1953, Noordhoff, Leyden, 1977.
- [7] M.M.S. Nasser, A boundary integral equation for conformal mapping of bounded multiply connected regions, *Comput. Methods Funct. Theory*, 9 (2009) 127-143.
- [8] M.M.S. Nasser, Numerical conformal mapping via boundary integral equations with the generalized Neumann kernels, *SIAM J. Sci. Comput.*, 31 (2009) 1695-1715.
- [9] M.M.S. Nasser, A.H.M. Murid and Z. Zamzami, A boundary integral method for the Riemann-Hilbert problem in domains with corners, *Complex Var. Elliptic Equ.*, 53 (11) (2008) 989-1008.
- [10] I.N. Vekua, *Generalized Analytic Functions*, Pergamon, London, 1992.
- [11] R. Wegmann, Constructive solution of a certain class of Riemann-Hilbert problems on multiply connected circular regions, *J. Comput. Appl. Math.* 130 (2001) 139-161.
- [12] R. Wegmann, A.H.M. Murid and M.M.S. Nasser, The Riemann-Hilbert problem and the generalized Neumann kernel, *J. Comput. Appl. Math.* 182 (2005) 388-415.
- [13] R. Wegmann and M.M.S. Nasser, The Riemann-Hilbert problem and the generalized Neumann kernel on multiply connected regions, *J. Comput. Appl. Math.* 214 (2008) 36-57.
- [14] G.C. Wen, *Conformal Mapping and Boundary Value Problems*, English translation of Chinese edition 1984, AMS, Providence, 1992.