**Steiner Distance Polynomial of Graph**

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**Abstract:** The steiner n-distance polynomial of a connected graph $G$, $W_n(G; x)$, is defined as $\sum_{k \geq n-1} M_n(G, k)x^k$ where $M_n(G, k)$ is the number of n-sets of vertices of $G$ that are of n-distance $k$. Such polynomials $W_n(G; x)$ are obtained for some special graphs and for compound graph $G_1 \bullet G_2$ and $G_1 : G_2$. Moreover, we give an upper bound for the average n-distance $\mu_n(G)$.

**1. Introduction**

In this paper, we consider finite connected graphs without loops or multiple edges. For undefined concepts and notations see [1,2].

Let $G = (V, E)$ be a connected $(p, q)$ graph, and let $S$ be an n-subset, $2 \leq n \leq p$, of vertices of $G$. The Steiner distance of $S$, denoted by $d_G(S)$, is the number of edges in a smallest connected sub-graph of $G$ containing S, called a Steiner tree. If $n = 2$, then the Steiner distance of $S$ is the known distance between two vertices of $S$. Steiner trees have applications to multiprocessor computer networks. For example, it may be desired to connect a certain set of processors with a sub-network that uses the smallest number of communication links. A Steiner tree for the vertices representing such processors that need to be connected corresponds to such a sub-network.

The *total Steiner distance* of a graph $G$, for $n \geq 2$, or *total Steiner n-distance* is denoted by $D_n(G)$, and defined as:

$$D_n(G) = \sum_{S \subseteq V} d_G(S)$$

where $|S| = n$. 
The average Steiner n-distance of $G$, $\mu_n(G)$, is defined as:

$$
\mu_n(G) = \left( \frac{p}{n} \right)^{-1} D_n(G)
$$

The Steiner n-diameter,

$$
diam_n(G) = \max_{S \subseteq V} d_G(S), \quad |S| = n.
$$

In 1997, P. Dankelmann, H. C. Swaet and O. R. Oellermann [3], studied the average Steiner n-distance and obtained upper and lower bounds for $\mu_n(G)$.

The Weiner polynomial or distance polynomial of a graph $G$ [4,5] is defined as:

$$
W(G; x) = \sum_{k=0}^{\delta} d(G, k)x^k
$$

in which $d(G, k)$ is the number of pairs of vertices of distance $k$, and $\delta$ is the diameter of $G$.

In this paper we study the Steiner distance polynomial of $G$, which we define in the following.

**Definition (1.1):** Let $G$ be a $(p, q)$ connected graph of the steiner n-diameter $\delta_n$ where $3 \leq n \leq p$. Then, the steiner n-distance polynomial of $G$ is defined as:

$$
W_n(G; x) = \sum_{k=\delta_n}^{\delta} M_n(G, k)x^k
$$

where $M_n(G, k)$ is the number of n-sets of vertices of $G$ that are of distance $k$.

It is clear that (2) is not exactly a generalization of (1); when $n = 2$, (1) gives:
\[ W_2(G; x) = W(G; x) - p \]  
(3)

One may easily see that:

\[ D_n(G) = \left. \frac{d}{dx} W_n(G; x) \right|_{x=1} = \sum_{k=1}^{\delta_n} k M_n(G, k) \]  
(4)

Thus, \( W_n(G; x) \) gives us \( \mu_n(G) \).

**Definition (1.2):** Let \( v \) be a vertex of a connected graph \( G \), and let \( 1 \leq n \leq \delta_n \), we define the polynomial:

\[ W_n(v, G; x) = \sum_{k=0}^{\delta_n} M_n(v, G; k)x^k \]  
(5)

where \( M_n(v, G; k) \) is the number of \( n \)-sets, \( 1 \leq n \leq p \), that contain vertex \( v \) and each of Steiner distance \( k \).

The number \( d_n(v, G) \) is defined in \([\ ]\) as:

\[ d_n(v, G) = \sum_{v \in S} d_n(S) \]  
(6)

Thus:

\[ d_n(v, G) = \left. \frac{d}{dx} W_n(v, G; x) \right|_{x=1} \]  
(7)

**2. Steiner n-Distance Polynomial of Some Special Graphs**

We give \( W_n(G; x) \) when \( G \) is a special graph such as complete graph \( K_p \), bipartite complete graph \( K_{p_1,p_2} \), a star graph \( S_p \), wheel graph \( W_p \), and a path graph \( P_p \), and then deduce \( \mu_n(G) \) for each such graph.
Theorem 2.1: For each $3 \leq n \leq p$, we have:

1) $W_n(K_p; x) = \left( \begin{array}{c} p \\ n \end{array} \right) x^{n-1}$

2) $W_n(K_{p_1,p_2}; x) = \left[ \left( \begin{array}{c} p_1 \\ n \end{array} \right) + \left( \begin{array}{c} p_2 \\ n \end{array} \right) \right] x^n + \left[ \sum_{r=1}^{n-1} \left( \begin{array}{c} p_1 \\ r \end{array} \right) \left( \begin{array}{c} p_2 \\ n-r \end{array} \right) \right] x^{n-1}$

3) $W_n(S_p; x) = \left( \begin{array}{c} p-1 \\ n-1 \end{array} \right) x^{n-1} + \left( \begin{array}{c} p-1 \\ n \end{array} \right) x^n$

4) $W_n(W_p; x) = \left[ \left( \begin{array}{c} p-1 \\ n-1 \end{array} \right) + \left( \begin{array}{c} p-1 \\ n \end{array} \right) \right] x^{n-1} + \left[ \left( \begin{array}{c} p-1 \\ n \end{array} \right) - (p-1) \right] x^n$

with the assumption that $\left( \begin{array}{c} a \\ b \end{array} \right) = 0$ whenever $a < b$.

Proof: One can easily prove $W_n(G; x)$ for each such special graphs by calculating $M_n(G, k)$ for $k = n - 1$ and for $k = n$ only.

Using theorem 2.1 with (4) we obtain the following result:

Corollary 2.2: For each of $3 \leq n \leq p$, we have:

1) $\mu_n(K_p) = n - 1$

2) $\mu_n(K_{p_1,p_2}) = \left( \begin{array}{c} p_1 + p_2 \\ n \end{array} \right)^{-1} \left( \left( \begin{array}{c} p_1 \\ n \end{array} \right) + \left( \begin{array}{c} p_2 \\ n \end{array} \right) + (n-1) \sum_{r=1}^{n-1} \left( \begin{array}{c} p_1 \\ r \end{array} \right) \left( \begin{array}{c} p_2 \\ n-r \end{array} \right) \right)$

3) $\mu_n(S_p) = n - \frac{n}{p}$

4) $\mu_n(W_p) = n - \frac{n}{p} - (p-1)\left( \begin{array}{c} p \\ n \end{array} \right)^{-1}$
The next theorem gives us the Steiner n-distance polynomial of a path graph.

**Theorem 2.3:** Let $P_p$ be a path graph of order $p$ and let $3 \leq n \leq p$, then:

$$W_n(P_p; x) = \sum_{k=n-1}^{p-1} (p-k) \binom{k-1}{k+1-n} x^k$$

(8)

**Proof:** It is clear that for every subset $S \subseteq V(P_p)$, $|S| = n$, the Steiner tree for $S$ is a subpath of $P_p$. The $n$-diameter of $P_p$ is $p - 1$. Let $P_p$ be as in Fig. 1.

```
\begin{center}
\begin{tikzpicture}
\node (v1) at (0,0) {$v_1$};
\node (v2) at (1,0) {$v_2$};
\node (v3) at (2,0) {$v_3$};
\node (v4) at (3,0) {$v_{p-1}$};
\node (vp) at (4,0) {$v_p$};
\draw (v1) -- (v2) -- (v3) -- (v4) -- (vp);
\end{tikzpicture}
\end{center}
```

Then:

$$M_n(P_p, n-1) = [(p-(n-1))] \binom{n-2}{0},$$

$$M_n(P_p, n) = (p-n) \binom{n-1}{1}$$

$$\vdots$$

$$\therefore M_n(P_p, k) = (p-k) \binom{k-1}{k+1-n}, \text{ for } k = n-1, n, \ldots, p-1$$

This is because if $R$ is a subpath of length $k$ with its terminals in $S$, then we have to choose $(k-1)-(n-2)$ vertices from $k-1$ vertices to be in $S$ for such $R$. The no. of such $R$ subpaths is $p-k$. Since:
\[ W_n(P_p;x) = \sum_{k=n-1}^{p-1} M_n(P_p,k), \text{ then we have the required formula (8).} \]

From theorem 2.3, we obtain \( D_n(P_p) \) and \( \mu_n(P_p) \) as stated in the following result:

**Corollary 2.4**: For \( 3 \leq n \leq p-1 \), and for every path graph \( P_p \), we have:

\[
\mu_n(P_p) = \frac{1}{(n-2)! \binom{p}{n}} \sum_{k=n-1}^{p-1} \frac{(p-k)\, k!}{(k+1-n)!} \tag{9}
\]

\[
= \left[ (n-2)! \binom{p}{n} \right]^{-1} \sum_{k=n-1}^{p-1} [(p-k)k(k-1)\cdots(k-n+2)]
\]

One may easily find that \( \mu_3(P_p) = \frac{1}{2}(p+1) \).

It is clear that if \( T \) is a spanning tree of a connected graph \( G \) of order \( p \), then:

\[
\mu_n(G) \leq \mu_n(T) \quad \text{for each } 2 \leq n \leq p-1. \tag{10}
\]

Moreover, if \( P_p \) is a path graph then:

\[
\mu_n(T) \leq \mu_n(P_p) \tag{11}
\]

Therefore, we have from corollary 2.4.

**Corollary 2.5**: For any connected graph \( G \) of order \( p \) and for every \( 3 \leq n \leq p-1 \), we have:

\[
\mu_n(G) \leq \left[ (n-2)! \binom{p}{n} \right]^{-1} \sum_{k=n-1}^{p-1} [(p-k)k(k-1)\cdots(k-n+2)] \tag{12}
\]
Equality holds if and only if $G = P_p$.
The above result gives an upper bound for the average Steiner n-distance for $n = 3$, $\mu_3(G) \leq \frac{1}{2}(p + 1)$.
The following corollary is needed in the next section.

**Corollary 2.6:** Let $\nu$ be a terminal vertex of the path graph $P_p$, and let $2 \leq n \leq p$. Then:

$$W_n(\nu, P_p; x) = \sum_{k=n-1}^{p-1} \binom{k - 1}{k + 1 - n} x^k$$  \hspace{1cm} (13)

**Proof:** It is clear that any $n$-set $S$ of vertices in $P_p$ either contains $\nu$ or it is a subset of $P_{p-1}$ obtained from $P_p$ by deleting vertex $\nu$.

Thus:

$$W_n(\nu, P_p; x) = W_n(P_p; x) - W_n(P_{p-1}; x)$$

$$= \sum_{k=n-1}^{p-1} (p - k) \binom{k - 1}{k + 1 - n} x^k - \sum_{k=n-1}^{p-2} (p - k - 1) \binom{k - 1}{k + 1 - n} x^k$$

Simplifying the summations we get the required result. \hfill \Box

3. Steiner n-Distance Polynomial of Compound Graphs

Let $G_1$ and $G_2$ be vertex-disjoint connected graphs, and let $u \in V(G_1)$ and $v \in V(G_2)$. Then, the graph $G_1 \bullet G_2$ defined by Gutman [4] as the compound graph obtained from $G_1$ and $G_2$ by identifying the two vertices $u$ and $v$.

Moreover, Gutman defined the compound graph $G_1; G_2$ as the graph obtained from $G_1$ and $G_2$ by joining the two vertices $u$ and $v$ by an edge. The Wiener polynomials of $G_1 \bullet G_2$ and $G_1; G_2$ are given by Gutman as:
\[W(G_1 \bullet G_2; x) = W(G_1; x) + W(G_2; x) + W(u, G_1; x)W(v, G_2; x) - W(u, G_1; x) - W(v, G_2; x)\]  \hspace{1cm} (14)

\[W(G_1; x) = W(G_1; x) + W(G_2; x) + xW(u, G_1; x)W(v, G_2; x)\]  \hspace{1cm} (15)

In this section, we obtain the Steiner n-distance polynomials of \(G_1 \bullet G_2\) and \(G_1 : G_2\); and then use that to find an upper bound for \(\mu_n(G)\).

First, we start with the following simple result:

**Theorem 3.1:** Let \(G_1 + G_2\) be the join of the disjoint connected graphs \(G_1\) and \(G_2\) of orders \(p_1\) and \(p_2\) respectively, Then:

\[W_n(G_1 + G_2; x) = Ax^n + Bx^{n-1}\]  \hspace{1cm} (16)

where:

\[A = \binom{p_1}{n} + \binom{p_2}{n} - M_n(G_1, n-1) - M_n(G_2, n-1),\]

\[B = \binom{p_1 + p_2}{n} - A\]

**Proof:** Let \(S\) be any n-set vertices of \(G_1 + G_2\). Then

i) If \(S \cap V(G_i) \neq \phi\) for \(i = 1\) and \(2\), then \(d_{G_1 + G_2}(S) = n - 1\).

ii) If for \(i = 1, 2\), \(S \subset V(G_i)\), Then :

\[d_{G_i + G_2}(S) = \begin{cases} n - 1 & , \text{ when } S \text{ is connected in } G_i \\ n & , \text{ when } S \text{ is disconnected in } G_i \end{cases}\]

Thus:
\[ M_n(G_1 + G_2, n-1) = M_n(G_1, n-1) + M_n(G_2, n-1) + \sum_{r=1}^{\frac{n}{2}} \binom{p_1}{r} \binom{p_2}{n-r}, \]

\[ M_n(G_1 + G_2, n) = \left( \frac{p_1}{n} \right) + \left( \frac{p_2}{n} \right) - M_n(G_1, n-1) - M_n(G_2, n-1) \]

since,

\[ \left( \frac{p_1 + p_2}{n} \right) = \sum_{k \neq 1} M_n(G_1 + G_2, k) = M_n(G_1 + G_2, n) + M_n(G_1 + G_2, n-1) \]

Then by substituting we get the required formula for

\[ W_n(G_1 + G_2; x). \]

**Theorem 3.2:** For \( 3 \leq n \leq \delta_n(G_1 \bullet G_2) \), we have:

\[ W_n(G_1 \bullet G_2; x) = W_n(G_1; x) + W_n(G_2; x) + W_n(u, G_1; x)W_2(v, G_2; x) \]

\[ + \sum_{r=2}^{\frac{n}{2}} W_r(u, G_1; x) \left[ W_{r-1}(v, G_2; x) + W_{r-2}(v, G_2; x) \right] \]

**Proof:** In \( G_1 \bullet G_2 \), let \( w \) be the vertex obtained from identifying \( u \) and \( v \). Let \( S \) be an \( n \)-set of vertices of \( G_1 \bullet G_2 \). Then, we have the following cases:

i) If \( S \subset V(G_1) \) or \( S \subset V(G_2) \), then:

\[ d_{G_1 \bullet G_2}(S) = d_{G_1}(S) \text{ or } d_{G_2}(S), \text{ respectively.} \]

ii) If \( S \cap V(G_1) \neq \emptyset \), \( S \cap V(G_2) \neq \emptyset \) and \( w \in S \), then:

\[ d_{G_1 \bullet G_2}(S) = d_{G_1}(S_1) + d_{G_2}(S_2), \text{ where} \]

\( S_i = S \cap V(G_i) \) for \( i = 1, 2 \).

iii) If \( S \cap V(G_1) \neq \emptyset \), \( S \cap V(G_2) \neq \emptyset \) and \( w \not\in S \), then:

\[ d_{G_1 \bullet G_2}(S) = d_{G_1}(S_1') + d_{G_2}(S_2'), \text{ where } S_i' = S_i \cup \{w\} \]

for \( i = 1, 2 \).
From the above cases we deduce that for \( n \geq 3 \):

\[
M_n(G_1 \bullet G_2, k) = M_n(G_1, k) + M_n(G_2, k) + \sum_{j=2}^{n-1} \sum_{i=1}^{j-1} M_i(u, G_1, j) M_{n-i-j}(v, G_2, k-j)
\]

\[
+ \sum_{j=2}^{n-1} M_i(u, G_1, j) M_{n-i-j}(v, G_2, k-j)
\]

\[
= M_n(G_1, k) + M_n(G_2, k) + \sum_{j=2}^{n-1} \left( \sum_{i=1}^{j-1} M_i(u, G_1, j) \left[ M_{n-i-j}(v, G_2, k-j) + M_{n-j}(v, G_2, k-j) \right] \right)
\]

\[
+ \sum_{j=1}^{n-1} M_n(u, G_1, j) M_2(v, G_2, k-j)
\]

Hence, \( \sum_{k=n-1}^{\infty} M_n(G_1 \bullet G_2, k) x^k \) equals to the formula given in (17).

In [3], the graph \( H_{p,k} \), \( k < p \), is defined as the graph constructed from a complete graph of order \( k \) and a path graph of order \( p-k+1 \) by identifying a terminal \( v \) of the path graph \( P_{p-k+1} \) with a vertex \( u \) of the complete graph \( K_k \). That is:

\[
H_{p,k} = K_k \cdot P_{p-k+1}.
\]

Then it is proved [ ] that for any connected graph of order \( p \), \( 2 \leq n \leq p \) and chromatic number \( k \)

\[
\mu_n(G) \leq \mu_n(H_{p,k})
\]

with equality if and only if \( G = H_{p,k} \).

In order to find such upper bound in terms of \( p, k \) and \( n \), we use

**Theorem 3.2:** Taking \( G_2 = K_k \) and \( G_1 = P_{p-k+1} \), it is clear that:
\[ W_n(v, K_p; x) = \binom{p-1}{n-1} x^{n-1} \]

Thus, using theorems 2.3, 3.2 and corollary 2.6, we get:

\[ W_n(H_{p,k}; x) = W_n(P_{p-k+1}; x) + W_n(K_k; x) + W_n(u, P_{p-k+1}; x)W_2(v, K_k; x) \]

\[ + \sum_{r=2}^{n-k} W_r(u, P_{p-k+1}; x)[W_{n-r+1}(v, K_k; x) + W_{n-r+1}(v, K_k; x)] \]

\[ = \sum_{r=1}^{n-k} \left( p - k + 1 - h \right) \left( \begin{array}{c} h-1 \\ h+1-n \end{array} \right) x^h + \left( \begin{array}{c} k \\ n \end{array} \right) x^{n-1} \]

\[ + \left( \begin{array}{c} k \\ 2 \end{array} \right) \sum_{h=n-1}^{h+1} \left( \begin{array}{c} h-1 \\ h+1-n \end{array} \right) x^h \]

\[ + \sum_{r=2}^{n-k} \sum_{h=n-r}^{h+1} \left( \begin{array}{c} h+n-r \\ h+1-r \end{array} \right) x^h \left( \begin{array}{c} k-1 \\ n-r+1 \end{array} \right) x^{n-r} + \left( \begin{array}{c} k-1 \\ n-r+1 \end{array} \right) x^{n-r+1} \]

\[ \frac{d}{dx} W_n(H_{p,k}; x) = \sum_{h=n-1}^{h+1} h(p-k+1-h) \left( \begin{array}{c} h-1 \\ h+1-n \end{array} \right) + (n-1) \left( \begin{array}{c} k \\ n \end{array} \right) \]

\[ + \left( \begin{array}{c} k \\ 2 \end{array} \right) \sum_{h=n-1}^{h+1} \left( \begin{array}{c} h+1 \\ h+1-n \end{array} \right) \]

\[ + \sum_{r=2}^{n-k} \sum_{h=n-r}^{h+1} \left( \begin{array}{c} h+n-r \\ h+1-r \end{array} \right) \left( \begin{array}{c} k-1 \\ n-r+1 \end{array} \right) \]

\[ + \sum_{r=2}^{n-k} \sum_{h=n-r}^{h+1} \left( \begin{array}{c} h+n-r \\ h+1-r \end{array} \right) \left( \begin{array}{c} k \\ n-r+1 \end{array} \right) + \left( \begin{array}{c} k-1 \\ n-r+1 \end{array} \right) \]

\[ \therefore D_n(H_{p,k}) = (n-1) \left( \begin{array}{c} k \\ n \end{array} \right) + \sum_{h=n-1}^{h+1} \frac{1}{2} k(k-1)(h+1) + h(p-k+1-h) \left( \begin{array}{c} h-1 \\ h+1-n \end{array} \right) \]

\[ + \sum_{r=2}^{n-k} \sum_{h=n-r}^{h+1} \left( \begin{array}{c} h+n-r \\ h+1-r \end{array} \right) \left( \begin{array}{c} k \\ n-r+1 \end{array} \right) + \left( \begin{array}{c} k-1 \\ n-r+1 \end{array} \right) \]

It seems that formula (19) is complicated, but it is useful for given \( k \) and \( n \). For example, if \( G \) is a planar graph, then \( k \leq 4 \). If we assume that \( k = 4 \) and \( n = 3 \), we obtain:
\[ D_3(H_{p,t}) = 8 + \sum_{h=2}^{p-4} (h-1)(hp - h^2 + 3h + 6) \]
\[ + \sum_{h=1}^{p-4} \left[ \frac{h-1}{h-1} \frac{(h+1)}{2} \right] \]
\[ = 8 + \sum_{h=1}^{p-4} \left\{ -h^3 + (p+4)h^2 - (p-9)h + 3 \right\} \]
\[ = \frac{1}{12} \left( p^4 + 10p^3 - 145p^2 + 506p - 504 \right) \]

Thus:
\[ \mu_3(H_{p,t}) = \frac{p^4 + 10p^3 - 145p^2 + 506p + 504}{2p(p-1)(p-2)} \quad (20) \]

Hence, we have the following result:

**Corollary 3.3:** For any connected planar graph \( G \) of order \( p \):
\[ \mu_3(G) \leq \mu_3(H_{p,t}) \], given in (20).

Moreover, if \( G \) is any connected graph of order \( p \), and \( H \) is a spanning planar subgraph of \( G \), then \( \mu_n(G) \leq \mu_n(H) \). Thus:

**Corollary 3.4:** For any connected graph of order \( p \):
\[ \mu_3(G) \leq \mu_3(H_{p,t}). \]

Now, we consider the compound graph \( G_1 \cdot G_2 \) using a method similar to that used for finding \( W_n(G_1 \cdot G_2; x) \).

**Theorem 3.5:** Let \( G_1 \) and \( G_2 \) be vertex-disjoint connected graphs of orders \( p_1 \) and \( p_2 \) respectively, and let
$u \in V(G_1)$, $v \in V(G_2)$ and $3 \leq n \leq p = p_1 + p_2$ then:

$$W_n(G_1 \cup G_2; x) = W_n(G_1; x) + W_n(G_2; x)$$

$$+ x \sum_{r=1}^{n-1} \left[ W_r(u, G_1; x) + W_{r+1}(u, G_1; x) \right] \left[ W_{n-r}(v, G_2; x) + W_{n-r+1}(v, G_2; x) \right]$$

(21)

**Proof:** Considering the distance for any $n$-set of vertices of $G_1 \cup G_2$, we obtain:

$$M_n(G_1 \cup G_2, k) = M_n(G_1, k) + M_n(G_2, k)$$

$$+ \sum_{r=1}^{n-1} \sum_{t=0}^{k-1} M_r(u, G_1, t) M_{n-r}(v, G_2, k-1-t)$$

$$+ \sum_{r=1}^{n-1} \sum_{t=0}^{k-1} M_r(u, G_1, t) M_{n-r+1}(v, G_2, k-1-t)$$

$$+ \sum_{r=1}^{n-1} \sum_{t=0}^{k-1} M_{r+1}(u, G_1, t) M_{n-r}(v, G_2, k-1-t)$$

$$+ \sum_{r=1}^{n-1} \sum_{t=0}^{k-1} M_{r+1}(u, G_1, t) M_{n-r+1}(v, G_2, k-1-t)$$

Thus, substituting in $\sum_{k=n-1}^{\delta} M_n(G_1 \cup G_2; k)x^k$, and noticing that in the last four double summations $x^k$ is written as $x x^t x^{k-1-t}$, we obtain the required formulas (21).

**References**


