STEINER DISTANCE POLYNOMIAL OF GRAPH

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Abstract: The steiner n-distance polynomial of a connected graph G, $W_n(G;x)$, is defined as $\sum_{k\geq n-1} M_n(G,k)x^k$ where $M_n(G,k)$ is the number of n-sets of vertices of G that are of n-distance k. Such polynomials $W_n(G;x)$ are obtained for some special graphs and for compound graph $G_1 \bullet G_2$ and $G_1 : G_2$. Moreover, we give an upper bound for the average n-distance $\mu_n(G)$.

1. Introduction

In this paper, we consider finite connected graphs without loops or multiple edges. For undefined concepts and notations see [1,2].

Let $G=(V,E)$ be a connected (p,q) graph, and let S be an n-subset, $2 \leq n \leq p$, of vertices of G. The Steiner distance of S, denoted by $d_G(S)$, is the number of edges in a smallest connected sub-graph of G containing S, *called a Steiner tree*. If $n=2$, then the Steiner distance of S is the known distance between two vertices of S. Steiner trees have applications to multiprocessors computer networks. For example, it may be desired to connect a certain set of processors with a sub-network that uses the smallest number of communication links. A Steiner tree for the vertices representing such processors that need to be connected corresponds to such a sub-network.

The total Steiner distance of a graph G, for $n \geq 2$, or *total Steiner n-distance* is denoted by $D_n(G)$, and defined as :

$$D_n(G) = \sum_{S\subseteq V} d_G(S) \qquad , \qquad |S| = n .$$

The average Steiner n-distance of G, $\mu_n(G)$, is defined as:

$$\mu_n(G) = \binom{p}{n}^{-1} D_n(G)$$

The Steiner n-diameter,

$$diam_n(G) = \max_{S \subseteq V} d_G(S) \qquad , \qquad |S| = n.$$

In 1997, P. Dankelmann, H. C. Swaet and O. R. Oellermann [3], studied the average Steiner n-distance and obtained upper and lower bounds for $\mu_n(G)$.

The Weiner polynomial or distance polynomial of a graph G [4,5] is defined as:

$$W(G;x) = \sum_{k=0}^{\delta} d(G,k)x^k \tag{1}$$

in which $d(G,k)$ is the number of pairs of vertices of distance k, and δ is the diameter of G.

In this paper we study the Steiner distance polynomial of G, which we define in the following.

Definition (1.1): Let G be a (p, q) connected graph of the steiner n-diameter δ_n where $3 \le n \le p$. Then, the steiner n-distance polynomial of G is defined as:

$$W_n(G;x) = \sum_{k=n-1}^{\delta_n} M_n(G,k)x^k \tag{2}$$

where $M_n(G,k)$ is the number of n-sets of vertices of G that are of distance k.

It is clear that (2) is not exactly a generalization of (1); when $n = 2$, (1) gives:

$$W_2(G;x) = W(G;x) - p \qquad (3)$$

One may easily see that:

$$D_n(G) = \frac{d}{dx} W_n(G;x) \Big]_{x=1} = \sum_{k=n-1}^{\delta_n} k M_n(G,k) \qquad (4)$$

Thus, $W_n(G;x)$ gives us $\mu_n(G)$.

Definition (1.2): Let v be a vertex of a connected graph G, and let $1 \le n \le \delta_n$, we define the polynomial:

$$W_n(v,G;x) = \sum_{k=0}^{\delta_n} M_n(v,G;k) x^k \qquad (5)$$

where $M_n(v,G;k)$ is the number of n-sets, $1 \le n \le p$, that contain vertex v and each of Steiner distance k.
The number $d_n(v,G)$ is defined in [ ] as:

$$d_n(v,G) = \sum_{v \in S} d_n(S) \qquad (6)$$

Thus:

$$d_n(v,G) = \frac{d}{dx} W_n(v,G;x) \Big|_{x=1} \qquad (7)$$

2. Steiner n-Distance Polynomial of Some Special Graphs

We give $W_n(G;x)$ when G is a special graph such as complete graph K_p, bipartite complete graph K_{p_1,p_2}, a star graph S_p, wheel graph W_p, and a path graph P_p, and then deduce $\mu_n(G)$ for each such graph.

Theorem 2.1: For each $3 \le n \le p$, we have:

1) $W_n(K_p;x) = \binom{p}{n} x^{n-1}$

2) $W_n(K_{p_1,p_2};x) = \left[\binom{p_1}{n} + \binom{p_2}{n}\right] x^n + \left[\sum_{r=1}^{n-1} \binom{p_1}{r} \cdot \binom{p_2}{n-r}\right] x^{n-1}$

3) $W_n(S_p;x) = \binom{p-1}{n-1} x^{n-1} + \binom{p-1}{n} x^n$

4) $W_n(W_p;x) = \left[(p-1) + \binom{p-1}{n-1}\right] x^{n-1} + \left[\binom{p-1}{n} - (p-1)\right] x^n$

with the assumption that $\binom{a}{b} = 0$ whenever $a < b$.

Proof: One can easily prove $W_n(G;x)$ for each such special graphs by calculating $M_n(G,k)$ for $k = n-1$ and for [redacted] $= n$ only.

Using theorem 2.1 with (4) we obtain the following result:

Corollary 2.2: For each of $3 \le n \le p$, we have:

1) $\mu_n(K_p) = n-1$

2) $\mu_n(K_{p_1,p_2}) = \binom{p_1+p_2}{n}^{-1} \left\{ n\binom{p_1}{n} + n\binom{p_2}{n} + (n-1)\sum_{r=1}^{n-1} \binom{p_1}{r}\binom{p_2}{n-r} \right\}$

3) $\mu_n(S_p) = n - \frac{n}{p}$

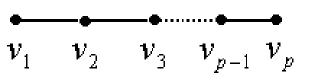
4) $\mu_n(W_n) = n - \frac{n}{p} - (p-1)\binom{p}{n}^{-1}$

The next theorem gives us the Steiner n-distance polynomial of a path graph.

Theorem 2.3: Let P_p be a path graph of order p and let $3 \le n \le p$, then:

$$W_n(P_p;x) = \sum_{k=n-1}^{p-1} (p-k)\binom{k-1}{k+1-n} x^k \tag{8}$$

Proof: It is clear that for every subset $S \subset V(P_p)$, $|S| = n$, the Steiner tree for S is a subpath of P_p. The n-diameter of P_p is $p-1$. Let P_p be as in Fig. 1.



Then:

$$M_n(P_p, n-1) = [p-(n-1)] \binom{n-2}{0} ,$$

$$M_n(P_p, n) = (p-n) \binom{n-1}{1}$$

$$\vdots$$

$$\therefore\ M_n(P_p, k) = (p-k) \binom{k-1}{k+1-n}, \text{ for}$$

$k = n-1, n, \cdots, p-1$

This is because if R is a subpath of length k with its terminals in S, then we have to choose $(k-1)-(n-2)$ vertices from $k-1$ vertices to be in S for such R. The no. of such R subpaths is $p-k$. Since:

$W_n(P_p;x)=\sum_{k=n-1}^{p-1} M_n(P_p,k)$, then we have the required formula (8). ∎

From theorem 2.3. we obtain $D_n(P_p)$ and $\mu_n(P_p)$ as stated in the following result:

Corollary 2.4: For $3 \le n \le p-1$, and for every path graph P_p, we have:

$$\mu_n(P_p)=\frac{1}{(n-2)!\binom{p}{n}}\sum_{k=n-1}^{p-1}(p-k)\frac{k!}{(k+1-n)!} \tag{9}$$
$$=\left[(n-2)!\binom{p}{n}\right]^{-1}\sum_{k=n-1}^{p-1}[(p-k)k(k-1)\cdots(k-n+2)]$$

One may easily find that $\mu_3(P_p)=\frac{1}{2}(p+1)$.

It is clear that if T is a spanning tree of a connected graph G of order p, then:

$$\mu_n(G)\le \mu_n(T) \qquad \text{for each } 2\le n\le p-1. \tag{10}$$

Moreover, if P_p is a path graph then:

$$\mu_n(T)\le \mu_n(P_p) \tag{11}$$

Therefore, we have from corollary 2.4.,

Corollary 2.5: For any connected graph G of order p and for every $3 \le n \le p-1$, we have:

$$\mu_n(G)\le\left[(n-2)!\binom{p}{n}\right]^{-1}\sum_{k=n-1}^{p-1}[(p-k)k(k-1)\cdots(k-n+2)] \tag{12}$$

Equality holds if and only if $G = P_p$.
The above result gives an upper bound for the average Steiner n-distance for $n = 3$, $\mu_3(G) \le \frac{1}{2}(p+1)$.
The following corollary is needed in the next section.

Corollary 2.6: Let v be a terminal vertex of the path graph P_p, and let $2 \le n \le p$. Then:

$$W_n(v, P_p; x) = \sum_{k=n-1}^{p-1} \binom{k-1}{k+1-n} x^k \tag{13}$$

Proof: It is clear that any n-set S of vertices in P_p either contains v or it is a subset of P_{p-1} abtained from P_p by deleting vertex v.
Thus:

$$W_n(v, P_p; x) = W_n(P_p; x) - W_n(P_{p-1}; x)$$
$$= \sum_{k=n-1}^{p-1} (p-k)\binom{k-1}{k+1-n} x^k - \sum_{k=n-1}^{p-2} (p-k-1)\binom{k-1}{k+1-n} x^k$$

Simplifying the summations we get the required result. ∎

3. Steiner n-Distance Polynomial of Compound Graphs

Let G_1 and G_2 be vertex-disjoint connected graphs, and let $u \in V(G_1)$ and $v \in V(G_2)$. Then, the graph $G_1 \bullet G_2$ defined by Gutman [4] as the compound graph obtained from G_1 and G_2 by identifying the two vertices u and v.

Moreover, Gutman defined the compound graph $G_1 : G_2$ as the graph obtained from G_1 and G_2 by joining the two vertices u and v by an edge. The Wiener polynomials of $G_1 \bullet G_2$ and $G_1 : G_2$ are given by Gutman as:

$$W(G_1 \bullet G_2;x) = W(G_1;x) + W(G_2;x) + W(u,G_1;x)W(v,G_2;x) - W(u,G_1;x) - W(v,G_2;x) \quad (14)$$

$$W(G_1 : G_2;x) = W(G_1;x) + W(G_2;x) + xW(u,G_1;x)W(v,G_2;x) \quad (15)$$

In this section, we obtain the Steiner n-distance polynomials of $G_1 \bullet G_2$ and $G_1 : G_2$; and then use that to find an upper bound for $\mu_n(G)$.

First, we start with the following simple result:

Theorem 3.1: Let $G_1 + G_2$ be the join of the disjoint connected graphs G_1 and G_2 of orders p_1 and p_2 respectively, Then:

$$W_n(G_1 + G_2;x) = Ax^n + Bx^{n-1} \quad (16)$$

where:

$$A = \binom{p_1}{n} + \binom{p_2}{n} - M_n(G_1, n-1) - M_n(G_2, n-1),$$

$$B = \binom{p_1 + p_2}{n} - A$$

Proof: Let S be any n-set vertices of $G + G_2$. Then

i) If $S \cap V(G_i) \neq \phi$ for $i = 1 \; and \; 2$, then $d_{G_1+G_2}(S) = n-1$.

ii) If for $i = 1,2$, $S \subset V(G_i)$, Then :

$$d_{G_1+G_2}(S) = \begin{cases} n-1 & , \quad when < S > is\ connected\ in\ G_i \\ n & , \quad when < S > is\ disconnected\ in\ G_i \end{cases}$$

Thus:

$$M_n(G_1+G_2,n-1)=M_n(G_1,n-1)+M_n(G_2,n-1)+\sum_{r=1}^{n-1}\binom{p_1}{r}\binom{p_2}{n-r},$$

$$M_n(G_1+G_2,n)=\binom{p_1}{n}+\binom{p_2}{n}-M_n(G_1,n-1)-M_n(G_2,n-1)$$

since,

$$\binom{p_1+p_2}{n}=\sum_{k\geq n-1}M_n(G_1+G_2,k)=M_n(G_1+G_2,n)+M_n(G_1+G_2,n-1)$$

Then by substituting we get the required formula for $W_n(G_1+G_2;x)$.

Theorem 3.2: For $3\leq n\leq \delta_n(G_1\bullet G_2)$, we have:

$$\begin{aligned}W_n(G_1\bullet G_2;x)&=W_n(G_1;x)+W_n(G_2;x)+W_n(u,G_1;x)W_2(v,G_2;x)\\&+\sum_{r=2}^{n-1}Wr(u,G_1;x)\left[W_{n-r+1}(v,G_2;x)+W_{n-r+2}(v,G_2;x)\right]\end{aligned} \quad (17)$$

Proof: In $G_1\bullet G_2$, let w be the vertex obtained from identifying u and v. Let S be an n-set of vertices of $G_1\bullet G_2$. Then, we have the following cases:

i) If $S\subset V(G_1)$ or $S\subset V(G_2)$, then:
$d_{G_1\bullet G_2}(S)=d_{G_1}(S)$ or $d_{G_2}(S)$, respectively.

ii) If $S\bigcap V(G_1)\neq\phi$, $S\bigcap V(G\ )\neq\phi$ and $w\in S$, then:
$d_{G\bullet G_2}(S)=d_{G}(S_1)+d_{G_2}(S_2)$, where
$S_i=S\bigcap V(G_i)$ for $i=1,2$.

iii) If $S\bigcap V(G_1)\neq\phi$, $S\bigcap V(G_2)\neq\phi$ and $w\notin S$, then :
$d_{G\bullet G_2}(S)=d_{G}(S_1')+d_{G_2}(S_2')$, where $S_i'=S_i\bigcup\{w\}$ for $i=1,2$.

From the above cases we deduce that for $n \geq 3$:

$$M_n(G_1 \bullet G_2,k) = M_n(G_1,k) + M_n(G_2,k) + \sum_{j=1}^{k-1}\sum_{i=2}^{n-1} M_i(u,G_1,j)M_{n+1-i}(v,G_2,k-j)$$

$$+\sum_{j=1}^{k-1}\sum_{i=2}^{n} M_i(u,G_1,j)M_{n+2-i}(v,G_2,k-j)$$

$$= M_n(G_1,k) + M_n(G_2,k)$$

$$+\sum_{i=2}^{n-1}\left\{\sum_{j=1}^{k-1} M_i(u,G_1,j)\left[M_{n+1-i}(v,G_2,k-j) + M_{n+2-i}(v,G_2,k-j)\right]\right\}$$

$$+\sum_{j=1}^{k-1} M_n(u,G_1,j)M_2(v,G_2,k-j)$$

Hence, $\sum_{k=n-1} M_n(G_1 \bullet G_2,k)x^k$ equals to the formula given in (17).

In [3], the graph $H_{p,k}$, $k < p$, is defined as the graph constructed from a complete graph of order k and a path graph of order $p-k+1$ by identifying a terminal v of the path graph P_{p-k+1} with a vertex u of the complete graph K_k. That is: $H_{p,k} = K_k \cdot P_{p-k+1}$.

Then it is proved [ ] that for any connected graph of order p , $2 \leq n \leq p$ and chromatic number k

$$\mu_n(G) \leq \mu_n(H_{p,k}) \tag{18}$$

with equality if and only if $G = H_{p,k}$.

In order to find such upper bound in terms of p, k and n, we use

Theorem 3.2: Taking $G_2 = K_k$ and $G \; = P_{p-k+}$. It is clear that:

$$W_n(v,K_p;x)=\binom{p-1}{n-1}x^{n-1}$$

Thus, using theorems 2.3, 3.2 and corollary 2.6, we get:

$$\begin{aligned}
W_n(H_{p,k};x) &= W_n(P_{p-k+1};x)+W_n(K_k;x)+W_n(u,P_{p-k+1};x)W_2(v,K_k;x)\\
&+\sum_{r=2}^{n-1}W_r(u,P_{p-k+1};x)\left[W_{n-r+1}(v,K_k;x)+W_{n-r+2}(v,K_k;x)\right]\\
&=\sum_{h=n-1}^{p-k}(p-k+1-h)\binom{h-1}{h+1-n}x^h+\binom{k}{n}x^{n-1}\\
&+\binom{k}{2}x\sum_{h=n-1}^{p-k}\binom{h-1}{h+1-n}x^h\\
&+\sum_{r=2}^{n-1}\left\{\sum_{h=r-1}^{p-k}\binom{h-1}{h+1-r}x^h\right\}\left[\binom{k-1}{n-r}x^{n-r}+\binom{k-1}{n-r+1}x^{n-r+1}\right]
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dx}W_n(H_{p,k};x)\bigg|_{x=1} &=\sum_{h=n-1}^{p-k}h(p-k+1-h)\binom{h-1}{h+1-n}+(n-1)\binom{k}{n}\\
&+\binom{k}{2}\sum_{h=n-1}^{p-k}(h+1)\binom{h-1}{h+1-n}\\
&+\sum_{r=2}^{n-1}\sum_{h=r-1}^{p-k}\left\{(h+n-r)\binom{h-1}{h+1-r}\binom{k-1}{n-r}+(h+n-r+1)\binom{h-1}{h+1-r}\binom{k-1}{n-r+1}\right\}
\end{aligned}$$

$$\begin{aligned}
\therefore D_n(H_{p,k}) &=(n-1)\binom{k}{n}+\sum_{h=n-1}^{p-k}\left[\frac{1}{2}k(k-1)(h+1)+h(p-k+1-h)\right]\binom{h-1}{h+1-n} \qquad (19)\\
&+\sum_{r=2}^{n-1}\sum_{h=r-1}^{p-k}\binom{h-1}{h+1-r}\left[(h+n-r)\binom{k}{n-r+1}+\binom{k-1}{n-r+1}\right]
\end{aligned}$$

It seems that formula (19) is complicated, but it is useful for given k and n. For example, if G is a planar graph, then $k \le 4$. If we assume that $k=4$ and $n=3$, we obtain:

$$D_3(H_{p,4}) = 8 + \sum_{h=2}^{p-4}(h-1)(hp - h^2 + 3h + 6)$$
$$+ \sum_{h=1}^{p-4}\binom{h-1}{h-1}\left[(h+1)\binom{4}{2} + \binom{3}{2}\right]$$
$$= 8 + \sum_{h=1}^{p-4}\left\{-h^3 + (p+4)h^2 - (p-9)h + 3\right\}$$
$$= \frac{1}{12}(p^4 + 10p^3 - 145p^2 + 506p - 504)$$

Thus:

$$\mu_3(H_{p,4}) = \frac{p^4 + 10p^3 - 145p^2 + 506p + 504}{2p(p-1)(p-2)} \tag{20}$$

Hence, we have the following result:

Corollary 3.3: For any connected planar graph G of order p:

$$\mu_3(G) \le \mu_3(H_{p,4}), \text{ given in (20).}$$

Moreover, if G is any connected graph of order p, and H is a spanning planar subgraph of G, then $\mu_n(G) \le \mu_n(H)$. Thus:

Corollary 3.4: For any connected graph of order p:

$$\mu_3(G) \le \mu_3(H_{p,4}).$$

Now, we consider the compound graph $G_1 : G_2$ using a method similar to that used for finding $W_n(G_1 \bullet G_2; x)$.

Theorem 3.5: Let G_1 and G_2 be vertex-disjoint connected graphs of orders p_1 and p_2 respectively, and let

$u \in V(G_1)$, $v \in V(G_2)$ and $3 \le n \le p = p_1 + p_2$ then:
$W_n(G_1 : G_2 ; x) = W_n(G_1; x) + W_n(G_2 ; x)$

$$+ x\sum_{r=1}^{n-1}\left[W_r(u,G_1;x) + W_{r+1}(u,G_1;x)\right]\left[W_{n-r}(v,G_2;x) + W_{n-r+1}(v,G_2;x)\right] \quad (21)$$

Proof: Considering the distance for any n-set of vertices of $G_1 : G_2$ as we have done for $G_1 \bullet G_2$, we obtain:

$$M_n(G_1 : G_2, k) = M_n(G_1, k) + M_n(G_2, k)$$

$$+ \sum_{r=1}^{n-1}\sum_{t=0}^{k-1} M_r(u, G_1, t) M_{n-r}(v, G_2, k-1-t)$$

$$+ \sum_{r=1}^{n-1}\sum_{t=0}^{k-1} M_r(u, G_1, t) M_{n-r+1}(v, G_2, k-1-t)$$

$$+ \sum_{r=1}^{n-1}\sum_{t=0}^{k-1} M_{r+1}(u, G_1, t) M_{n-r}(v, G_2, k-1-t)$$

$$+ \sum_{r=1}^{n-1}\sum_{t=0}^{k-1} M_{r+1}(u, G_1, t) M_{n-r+1}(v, G_2, k-1-t)$$

Thus, substituting in $\sum_{k=n-1}^{\delta_n} M_n(G_1 : G_2; k) x^k$, and noticing that in the last four double summations x^k is written as $x\, x^t\, x^{k-1-t}$, we obtain the required formulas (21).

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