Complete Arcs in a Projective Plane over Galois Field

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Abstract:

A $(k,n)-arc$ in $PG(2,p)$ is a set of k points no $n+1$ of which are collinear. A $(k,2)-arc$ is called $k-arc$ which is a set of k points where no three of them are collinear.

A $k-arc$ is complete if it is not contained in a $(k+1)-arc$. The maximum number of points that a $k-arc$ can have is $(p+1)$ for p odd or $(p+2)$ for p even. and $k-arc$ with this number of points is an oval.

Hirchfeld,1979 [4] showed the construction and classification of k-arcs over Galois field with $p \le 9$, and Rania, 1997[7] gave the construction and classification of $k-arc$ in $PG(2,11)$ over $G(11)$.

The aim of the present research is to find a way to add a point to a $k-arc$ in a projective plane $PG(2,p)$ Over Galois field $G(p)$ with p is odd number so that it keeps $k-arc$ subject to addition of more points until we get maximum complete arc which is an oval. We have found that at the beginning with $4-arc$, we can then add any point of the index zero. The choice of the fifth point determines the method of choosing the other points, because $4-arc$ with the fifth point represent a conic. In order that the sixth point is successfully chosen it must satisfies the conic equation. We have found $p+1$ conics of which $p-2$ non-degenerated in $PG(2,p)$, hence $p-2$ complete arcs (oval) through any $4-arc$ certainly there is only one conic which contains any $5-arc$.

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1-Projective Plane

Definition 1.1:

A projective plane PG(2,p) over GF(p) is a two –dimensional projective space, which consists of points and lines with incidents relations between them satisfy :

1. *Two points are contained one and only one line.*
2. *Two lines intersect in exactly one point.*
3. *There are four points such that no three of them are in the same line.*

In PG(2,p) there are 1+p+p2 points and 1+p+p2 lines, 1+p points on every line and 1+p lines through every point.

All these points in PG(2,p) have the form of a triple (x1,x2,x3) where x1,x2,x3 are elements in GF(p) with the exception of triple consisting of three zero elements.

Two triples (x1,x2,x3) and (y1,y2,y3) represent the same point if there exist $\lambda \in GF(p)\setminus\{0\}$*, such that (y1,y2,y3) =* λ *(x1,x2,x3).*

Finally the points of PG(2,p) can be numerated as follows:

The number of the point (1,0,0) is 1, the number of the point (x,1,0) is x+2.The point (x,y,1) is numerated as x+yp+p+2

Definition 1.2:

A polynomial F in $K[x_1, x_2, \ldots, x_n]$ *is called homogeneous or a form of degree d if all its terms have the same degree d.*

A sub set υ *of PG(n,K) is a variety (over K) if there exist forms F1,F2,...Fn in K[X], such that*

$$\upsilon = \{P(A) \in PG(n, K) / F_1(A) = F_2(A) = \ldots = F_r(A) = 0\} = V(F_1, F_2, \ldots, F_r)$$

The points P(A) are points of υ*. A variety V(F) in PG(n,K) is a primal. A primal in PG(2,K) is a plane (algebraic curve).*

The order or degree of a primal V(F) is the degree of F.

Definition 1.3:

A quadric Q in PG(n-1,p) is a primal of order two. So if Q is a quadric, then Q=V(H), where H is quadric form, that is

$$H=\sum_{\substack{i\le j\\ i,j=1}}^{n} a_{ij}x_ix_j = a_{11}x_1^2 + a_{12}x_1x_2 + \ldots + a_{nn}x_n^2$$

The form and the variety are degenerated if there is a change of coordinate system, which reduces the form to one in fewer variables. Otherwise, the form and the variety are non-degenerate.

Theorem 1.1:

In PG(2,q) with q even, the nucleus of the conic :

$$V(a_{00}x_{02} + a_{00}x_{02} + a_{11}x_{12} + a_{22}x_{22} + a_{01}x_0x_1 + a_{02}x_0x_2 + a_{12}x_1x_2)\ is$$

$$P(a_{12}, a_{02}, a_{01})$$

2-Complete Arcs in a Projective Plane PG (2,p) Over Galois Field GF (p) with p odd

At the beginning with $4-arc$, we can then add any point of the points of the index zero. The choice of the fifth point determines the method of choosing the other points, because 4-arc with the fifth point represent a conic. In order that the sixth point is successfully chosen it must satisfies the conic equation. We have found p+1 conics of which p-2 non-degenerated in PG(2,p), hence p-2 complete arcs (oval) through any 4-arc certainly there is only one conic which contains any 5-arc.

With reference points the pencil of conics is

$$V(ax_0x_1 + bx_0x_2 + cx_1x_2) = 0 \qquad with\ a+b+c=0.$$

Which are p+1 conics three of theme are degenerated.

1- $V(x_0x_1 - x_1x_2) = 0$ implies that $x_1 = 0 \ \ or \ \ x_0 - x_2 = 0$

2- $V(x_0x_1 - x_0x_2) = 0$ implies that $x_0 = 0 \ \ or \ \ x_1 - x_2 = 0$

3- $V(x_0x_2 - x_1x_2) = 0$ implies that $x_2 = 0$ or $x_0 - x_1 = 0$

and there are p-2 non-degenerated conics.

The equation of conic can be written as

$$x_0x_1 + \alpha x_0x_2 + \beta x_1x_2 = 0 \qquad \ldots\ldots\ldots\ldots(1)$$

where $\alpha = \frac{b}{a}$, $\beta = \frac{c}{a}$ $\Rightarrow \beta = -(1+\alpha)$

then (1) can be written as:

$$x_0x_1 + \alpha x_0x_2 - (1+\alpha)x_1x_2 = 0$$

where $\alpha \neq 0$ or p-1 for if $\alpha = 0$ or p-1 we get a degenerated conic. i.e., α=1,2,3,…,p-2.

With any value for α there is a unique conic with p+1 points four of them are the reference and unit points, where the other points are $\left(\frac{(1+\alpha)j}{\alpha+j}, j, 1\right)$,

j=2,3,…,p-1 , α+j ≠ 0 (mod p).

Then $C_\alpha = \{1, 2, p+2, 2p+3, (1+\alpha)j(\alpha+j)^{-1} + pj + p + 2\}$,

$\alpha = 1,2,3,\ldots, p-2$. $j = 2,3,4,\ldots, p-1$,

$\alpha + j \neq 0 (\bmod\ p)$.

The coordinates of the points of the ovals of PG(2,p) through the reference and unit points are given in Table (2.1).

Table (2.1).

The coordinate of points of the ovals of PG(2,p)through the reference and unit points

C_α	2	3	4	...	p-1
c1	$2*2*3^{-1}$	$2*3*4^{-1}$	$2*4*5^{-1}$	...	$2(p-1)p^{-1}$
c2	$3*2*4^{-1}$	$3*3*5^{-1}$	$3*4*6^{-1}$	...	$3(p-1)(p+1)^{-1}$
c3	$4*2*5^{-1}$	$4*3*6^{-1}$	$4*4*7^{-1}$	...	$4(p-1)(p+2)^{-1}$
	...	...	...	...	...
cp-2	$(p-1)*2*p^{-1}$	$(p-1)*3*(p+1)^{-1}$	$(p-1)*4*(p+2)^{-1}$	...	$(p-1)(p-1)(2p-3)^{-1}$

3-Complete Arcs in a Projective Plane PG(2,q) Over Galois Field GF(q) With q Even as a Power to Prime Number

In case of q is an even number as a power of prime number there are q+2 complete arcs containing any 4-arc. Each of these arcs represents a conic through q+1 points of PG(2,q) plus the nucleus of this conic. The choice of the nucleus determines the complete oval where the nucleus is either one of the points of 4-arc or any point (α,β,1) of the points of the index zero of the 4-arc which has the property of $\alpha + \beta + 1 = 0$.

Every complete oval in PG(2,q) through the reference and unit points is a conic plus its nucleus .

The conics and their nucleus which form the complete ovals in PG(2,q) through the reference and unit points are given in Table (3.1)

Table (3.1)

The complete ovals in PG(2,q) through the reference and unit points

I	The Conic CI	The nucleus Pi of CI	The set Ci of the point of VI
1	$x_0^2 + x_1 x_2 = 0$	$(1,0,0)$	$2,q+2,2q+3,a^2+aq+q+2$ *where a=2,3,4,…,p-1*
2	$x_1^2 + x_0 x_2 = 0$	$(0,1,0)$	$1,q+2,2q+3,a+a^2q+q+2$ *where a=2,3,4,…,p-1*
3	$x_2^2 + x_0 x_1 = 0$	$(0,0,1)$	$1,2,2q+3,a+a^{-1}q+q+2$ *where a=2,3,4,…,p-1*
4	$x_0 x_1 + x_0 x_2 + x_1 x_2 = 0$	$(1,1,1)$	$1,2,q+2,a(a+1)^{-1}+aq+q+2$ *where a=2,3,4,…,p-1*
5	$x_0 x_1 + (1-2) x_0 x_2 + 2 x_1 x_2 = 0$	$(2,1-2,1)$	*1,2,q+2,2q+3 , 2a.((1-2)+a)-1+qa+q+2 where a=2,3,4,…,p-1, a≠1-2*
6	$x_0 x_1 + (1-3) x_0 x_2 + 3 x_1 x_2 = 0$	$(3,1-3,1)$	*1,2,q+2,2q+3 ,3a((1-3)+ a)-1+aq+q+2 where a=2,3,4,…,q-1, a≠1-3*
7	$x_0 x_1 + (1-4) x_0 x_2 + 4 x_1 x_2 = 0$	$(4,1-4,1)$	*1,2,q+2,2q+3 ,4a((1-4)+ a)-1+aq+q+2 where a=2,3,4,…,q-1,a≠1-4*
…	…	…	…
Q+2	$x_0 x_1 + (1-(q-2)) x_0 x_2 + (q-2) x_1 x_2 = 0$	$(q-2,1-(q-2),1)$	*1,2,q+2,2q+3 ,(q-2)a((1-(q-2))+a)1+aq+q+2 where a=2,3,4,…,q-1,a≠1-(q-2)*

Conclusion

We see that when P is odd There is not a complete p-arc in a projective plane PG(2,p), and Every maximum Complete arc is (p+1)-arc which is conic as oval. And k is not oval if and only if k<p.

On other hand when q is even as a power of prime number, There is not a complete q-arc. And every maximum complete arc is (q+2)-arc which is a conic plus its nucleus as complete oval. and if k is a complete k-arc, then k is not oval if and only if k< q.

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