Mathematical model of the dynamics (changes) of the
temporal structures using Fokker-Plank Equation

AHMED NOORI Al-Aloosy

النموذج الرياضي لتغيرات التراكيب
الوقتية باستخدام معادلة فوكر-بلانك

ملخص
من المعروف أن معادلة فوكر-بلانك تصف ديناميكيا (تغيرات) تعديلات الكثافة
الاحتمالية المناسبة، وهذا ما بين خلاصة فضاء الطور متعدد الأبعاد، والموديلات
الرياضية المماثلة للهياكل الوقتية، والتي يتناولها هذا البحث، تسمح لنا، تحليليا،
بتحديد تفضيلات فضاء الطور لهذه الهياكل (أو البنية) والتي يمكن توظيفها
لتقييم الأضطرابات والضوضاء الإشارات.

SUMMARY
It is known, that the Fokker-plank Equation (FPE) describes
dynamics (changes) of modifications of probabilities density in
many-dimensional phase space. The represented mathematical
models of dynamics of temporal structures, allow to determine
analytical modifications phase coordinates of these structures of
perturbations and noise operations.

Introduction:
Synchronization phenomena been a topic of scientific research
for many years\(^{1}\) and in many systems, ranging from physics to
biology\(^{2}\). Many different situation have been considered,
including synchronization of limit cycle oscillators\(^{3,4}\),
synchronization of chaotic systems\(^{5}\), partial (i.e., phase)
synchronization\(^{6}\), generalized synchronization\(^{7,8}\), synchronization

* Hodeidah University Education college Physics Department
of stochastic systems\(^{(9)}\), systems characterized by a purely temporal
dynamics. Synchronization phenomena have also been studied in
systems with spatial degrees of freedom in par synchronization
particular, synchronization of two spatiotemporally chaotic
fields\(^{(11,12)}\). In Refs.\(^{(11,12)}\) the two chaotic fields are taken as the
two independent components of a vector field, as for example, the
two polarization components of an electric field vector.
Synchronization is mediated by the dynamics of spatially localized
vectorial structures. In this paper we address the question of the
dynamical of temporal structures of perturbation and noise
operations. This regime is characterized by the fact that local
perturbations are advocated more rapidly than their rate of
spreading. Therefore, macroscopic patterns named noise-sustained
structures emerge in this regime if noise is present at all time.
Optical noise-sustained structures\(^{(13)}\) have been predicted in optical
parametric oscillators (OPO’s) in which the nonlinear crystal is
pumped by a field linearly polarized with frequency \(f_0\) (signal and
idler) are orthogonally polarized. Our mathematical models of
dynamics of temporal structures, allow determining analytical
modifications phase coordinates of these structures of perturbations
and noise operations.

**Theory:**

Let consider dynamics (changes) of a system; its behavior is
described by the following differential equation:

\[
\frac{dT_i}{dt} + f(T_1, T_2, ..., T_n) = \xi_i \quad (i = 1, 2, ..., n)
\]  

\(T_i\) –is the phase coordinates, for example, the realization of
research durations temporal intervals; \(f(\ldots)\) describes power
performances of a phase space that influencing the modification of
this time intervals, and \(\xi_i\) is a random disturbance, operate on the
given temporal structure and changes of phase coordinates of the
given temporal structure, or it may be the exterior noise with a
known matrix of spectral densities \(S = ||S_{ik}||\), where
(k=1,2,...,n). Further we consider, that initial values phase
coordinates \(T_1(0), T_2(0), ..., T_n(0)\) are considered casual and their
distribution characterizes a probability density\(^{(14)}\).
Mathematical model of the dynamics – Ahmed Noori

\[ P(T_1, T_2, \ldots, T_n, 0) = P_0(T_1, T_2, \ldots) \]  
(2)

Which should satisfy the condition of normalization?

\[ \int_0^\infty \int_0^\infty \int_0^\infty \cdots \int_0^\infty P_0 dT_1 dT_2 \cdots dT_n = 1 \]  
(3)

The equation FPE allows basically to place (install) dynamic linking of the following probability density \( P(T_1, T_2, \ldots, T_n) \) with a performance of a temporal structure \( F_i(T_1, T_2, \ldots, T_n) \) as follows\(^{(15)}\).

\[ \frac{\partial \ln(p)}{\partial t} + \sum_{i=1}^{n} \frac{\partial^2 \ln(p)}{\partial T_i^2} - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 \ln(p)}{\partial T_i \partial T_j} + \sum_{i=1}^{n} \frac{\partial f(T, t)}{\partial T_i} = 0 \]  
(4)

Under certain condition the durations of time intervals \( T_i \) can be proportional to performances of a phase space power, expressed through \( F_i \). If it is limited only to these superposition's and to neglect exterior perturbations and noise, the equation of dynamics (changes) can be noted as a system of two equations:

\[ \frac{\partial \ln(p(T, t))}{\partial t} - f(T, t) \frac{\partial \ln(p(T, t))}{\partial T} = \frac{\partial f(T, t)}{\partial T} \]  
(5)

\[ \frac{dT}{dt} + f(T, t) = 0 \]

If the distribution density of the durations of time intervals dose not vary in time, the first term in the equation (4) is equal to 0, and then we get:

\[ \frac{\partial}{\partial T} [\ln f(T, t) + \ln p(T)] = \frac{\partial}{\partial T} \ln[f(T, t) p(T)] = 0 \]  
(6)

where, \( \ln f(T, t) + \ln p(T) = \varphi(T) \)

The function \( \varphi(T) \) is determined from the entry conditions. For example, when, \( T = T_0 \)

\( f(T, t) = f_0(T_0, t) \), in particular, if
\[ \sum_{i=1}^{n} k_i(t) \quad - \text{degree function, } f_0(T_0, t) = \]

\[ \varphi(t) = \ln \left[ f_0(T_0, t) p(T_0) \right], \]

When performance of a temporal structure required, the time intervals will look like the following:

\[ f(T, t) = \left[ f_0(T_0, t) P(T_0)/P(T) \right] \quad (7) \]

Substituting expression (7) in (4) and taking this system rather T(t), we shall have the following equation for the temporal dynamics (changes) of intervals:

\[ \int_{0}^{\infty} p(T) dT = - P(T_0) \int_{0}^{\infty} f_0(T_0, t) dt + C_1 \quad (8) \]

Where \( C_1 \) is a constant of integration, which is defined from the entry conditions. For example, when \( t=t_0 \) and \( T=T_0 \), we obtain, that

\[ C_1 = \left[ \int_{0}^{\infty} P(T) dT \right]_{T=T_0} + P(T_0) \int_{0}^{\infty} f(T_0, t) dt \bigg|_{t=t_0} \quad (9) \]

In this case the probability distribution \( p(T, t) \) varies in time, and the common solution is possible to be as

\[ f(T, t) = \]

\[ \left[ 1/p(T, t) \right] \left[ f(T_0, t) p(T_0, t) - \int_{0}^{\infty} \frac{\partial p(T, t)}{\partial t} dT \bigg|_{T=T_0} + \int_{0}^{\infty} \frac{\partial p(T, t)}{\partial t} dT \right] \quad (10) \]
For the case of the normal law of distribution we have

\[ p(T) = \left[ \frac{1}{\sqrt{2\pi\sigma}} \right] \exp \left[ \frac{-1}{2} (T - m/\sigma)^2 \right], \]

where \( s \) and \( I \) are constants factors.

When \( p(T,t) = p(T) \) and \( f(T,t) = K_\sigma \) - do not vary in time, the performance of a temporal structure will look like the following:

\[ f(T) = K_n \exp \left[ \frac{1}{2} (T - m/\sigma)^2 \right], \]

\[ K_n = K_0 \exp \left[ \frac{1}{2} (T_0 - m/\sigma)^2 \right]. \]

For more spreaded case, when the distribution \( p(T) \) has asymmetry and kurtosis \( E_x \) (that is two model) we shall have, that \( p(T) \) is possible to present as a sum of two normal distributions

\[ p(T) = g_1 \left( \frac{1}{\sqrt{2\pi\sigma_1}} \right) \exp \left[ - \frac{1}{2} (T - m_1/\sigma_1)^2 \right] + \]

\[ g_2 \left( \frac{1}{\sqrt{2\pi\sigma_2}} \right) \exp \left[ - \frac{1}{2} (T - m_2/\sigma_2)^2 \right] \]

(12)

In this equation the factors \( g_1, g_2, \sigma_1 \) and \( \sigma_2 \) are determined by the numerical methods of smoothing of stochastic serieses for average values \( (m_1, m_2) \), and also for known values of first four moment of casual distribution. Then, with allowance for equation (12) of dynamics (changes) of a modification of temporary duration intervals \( T(t) \) is possible to be as

\[ \frac{dT}{dt} + \frac{K_\sigma p(T)}{g_1 \left( \frac{1}{\sqrt{2\pi\sigma_1}} \right) e^{-\frac{1}{2} (T - m_1/\sigma_1)^2} + g_2 \left( \frac{1}{\sqrt{2\pi\sigma_2}} \right) e^{-\frac{1}{2} (T - m_2/\sigma_2)^2}} \]

(13)
In specific case, when \( p(T) \) varies linearly with a modification of duration \( T(t) \), we shall receive the following expression for \( p(T) \):

\[
P(T) = IT + P
\]  

(14)

Where \( I \) and \( P \) are constants factors.

Then we shall have, that

\[
\frac{dT}{dt} + k_o(I_T + P)/(IT + P) = 0
\]  

(15)

Integrating this equation we get

\[
\frac{IT^2}{2} + PT = -k_o(I_T + P) t + C_2
\]  

(16)

\( C_2 \) is a new constant of integration. Equation (16) can be used for small time interval of linear approximation \( p(T) \) of an arbitrary type. Thus we can have analytical expression for an association \( T(t) \) as nonlinear function. It is necessary to mark what dynamic is (change) to investigate by a very large amount of time intervals on the basic solution of the equations FPE. the correlation between entering and exit pupils of a research system is not known\(^{(16)}\).

Allowance model for perturbation

Let's consider now dynamics of a temporal structure with allowance for perturbation and noise operations. Accordingly let

\[
\frac{1}{2} \left[ \frac{\partial^2 \text{Inp}}{\partial A^2} + \left( \frac{\partial \text{Inp}}{\partial A} \right)^2 \right] = Q(A) \quad ; \quad \frac{\partial \text{Inp}}{\partial A} = R(A)
\]
Then for \( p(A,t) = p(A) \) we have

\[
\frac{\partial f}{\partial A} + f R(A) = -Q(A)
\]

The solution form of such equation will be

\[
f(A,t) = e^{-\int_R e^{gA} dA} (C_3 - \int Q(A) e^{\int_R e^{gA} dA} dA)
\]

(17)

For \( Q = 0 \), we will have an obvious relation as in case of a lack of perturbation and noise

\[
f(A,t) = e^{\int_{A_0}^{A} \int_R e^{gA} dA} \frac{C_3}{p}, \text{ where } f(A_0) = C_3
\]

(18)

if \( A = A_0 \), \( f(A,t) = f(A_0, t) \)

\[
f(A,t) = f(A_0, t) p(A_0) / p(A)
\]

Substituting a value of \( Q(A) \) and \( R(A) \), in (17). We will have the following equation of dynamics (changes) of a temporal structure:

\[
\frac{dA}{dt} + \frac{1}{p} \left\{ C_3 - \frac{1}{2} \int Z \left[ \frac{\partial^2 \int p}{\partial A^2} + \left( \frac{\partial \int p}{\partial A} \right)^2 \right] \right\} p \, dA = \xi
\]

For a common case piecewise linear approximation \( p(A) \) we have for \( p(A) = mA + n \)
\[
\frac{dA}{dt} + \frac{C_3}{mA+n} = k \tag{19}
\]

Integrating this equation, we will get

\[
\frac{A}{k} + \frac{C_4}{k^2m} \ln(kmA + kn - C_4) = t + C_3 \tag{20}
\]

When \( k \)- value of perturbations is 0 at \( t < 0 \) we get

\[
C_4 = k_0 (MA_0 + n);
\]

\[
C_3 = \frac{A_0}{k} + \frac{C_4}{k^2m} \ln \left(kmA_0 + kn - C_4\right)
\]

For a numerical solution of this equation of dynamics it’s necessary to know a value of scale factor of perturbation \( T_0 \). Scale factor (correspondence between a unit of measurements of revolving action and unit of measurements \( A \) ) is defined on static performance, which divide on some linear plots (sites). If the distribution \( p(A) \) can be described by expression of an aspect

\[
P(A) = k_1 e^{m(A-a)}
\]

That

\[
f(A, t) = (C_5 / K_1 e^{m(A-a)}) - Sm / 2
\]

The equation of dynamics will look like the following

\[
\frac{dA}{dt} + Be^{-m(A-a)} - \frac{Sm}{2} = d
\]

Solution of this last equation is possible to be

\[
A(t) = \frac{1}{m} \ln \frac{1}{C_5} \left[e^{C_5m(t+C_6)} + B\right] + a
\]
integration. Thus, the represented above mathematical models of dynamics of temporal structures, allow to determine analytically modifications phase coordinates of these structures for perturbations and noise operations.

Conclusion:
It's found that the mathematical model of dynamic of temporal structures which by the relation

\[ A(t) = \frac{1}{m} \ln \left( \frac{1}{C_5} \left[ e^{C_5 m (t + C_6)} + B \right] \right) + a \]

allow to determine analytically modifications phase coordinates of those structures for perturbation and noise operations.
REFERENCES: